Global solutions for a hyperbolic model of multiphase flow

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joint work with

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Introduction

We consider a hyperbolic model for 1–D multiphase reactive flow

$$\begin{cases} v_t - u_x & = 0 \\ u_t + p(v,\lambda)_x & = 0 \\ \lambda_t & = \dfrac{\alpha}{\tau}(p - p_e)\lambda(\lambda - 1) \end{cases}$$

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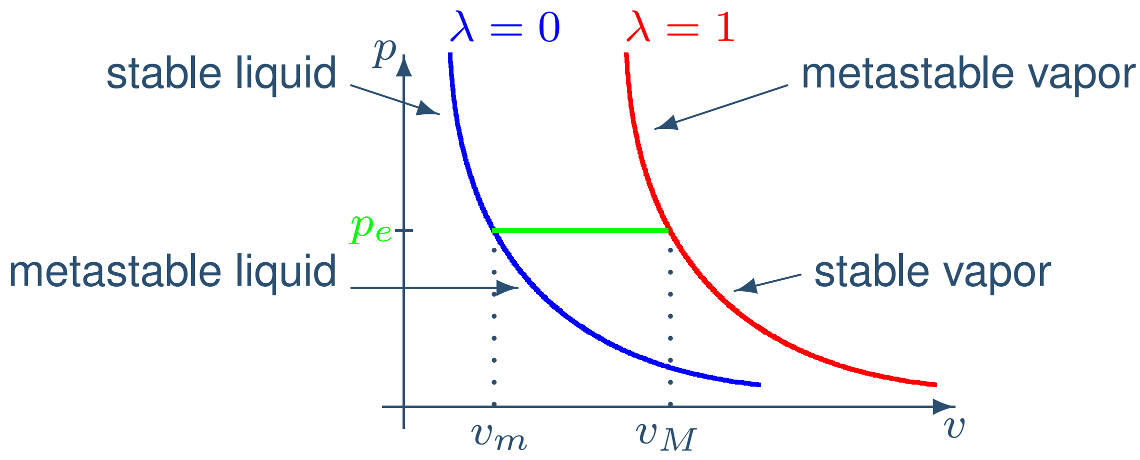
The model



$\lambda = 0$
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p
metastable vapor
p_e
metastable liquid
stable vapor
v_m
v_M
v

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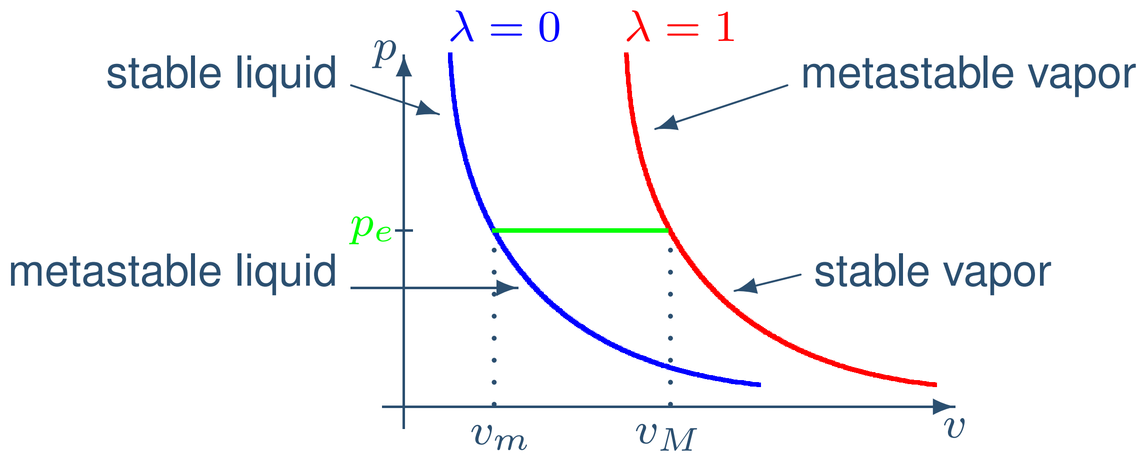

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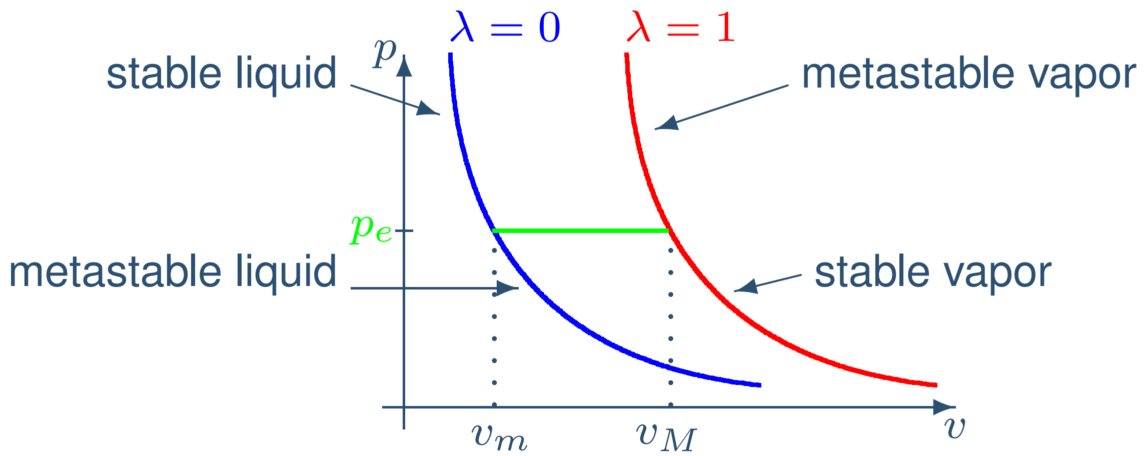

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The model may take into account viscosity terms

$$\begin{cases} v_t - u_x &= 0 \\ u_t + p_x &= \varepsilon u_{xx} \\ \lambda_t &= \dfrac{\alpha}{\tau}(p - p_e)\lambda(\lambda - 1) + b\varepsilon\lambda_{xx} \end{cases}$$

Riemann problem for the 0-viscosity and 0-relaxation limit: [Corli and Fan, 2005].

Aim of this work

- For a fixed relaxation time $\tau > 0$, look for global (in time) solutions of the Cauchy problem with large BV data

$$(v, u, \lambda)(0, x) = \left(v_o(x), u_o(x), \lambda_o(x)\right),$$

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- the homogeneous system: the 3×3 system of conservation laws

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We first focus on the analysis of system **(H)**.

Comparison with another model

In Eulerian coordinates, for $p = a^2(\lambda)\rho$ and $\rho = 1/v$, the system rewrites as

$$\begin{cases} \rho_t + (\rho u)_x & = 0, \\ (\rho u)_t + \big(\rho u^2 + p(\rho, \lambda)\big)_x & = 0, \\ (\rho\lambda)_t + (\rho\lambda u)_x & = 0\,. \end{cases}$$

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In [Benzoni-Gavage, 1991] many models for diphasic flows are proposed, for instance

$$\begin{cases} (\rho_l R_l)_t + (\rho_l R_l u_l)_x & = 0 \\ (\rho_g R_g)_t + (\rho_g R_g u_g)_x & = 0 \\ (\rho_l R_l u_l + \rho_g R_g u_g)_t + (\rho_l R_l u_l^2 + \rho_g R_g u_g^2 + p)_x & = 0\,. \end{cases}$$

Here l and g stand for *liquid* and *gas*; ρ_l, R_l, u_l are the liquid density, phase fraction, velocity, and analogously for the gas, $R_l + R_g = 1$, $p = a^2 \rho_g$.

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If $u_l = u_g$ and $\rho_l = 1$, define the concentration $c = \dfrac{\rho_g R_g}{\rho_l R_l}$ then [Peng, 1994]

$$\begin{cases} (R_l)_t + (R_l u)_x & = 0 \\ (R_l c)_t + (R_l c u)_x & = 0 \\ (R_l(1+c)u)_t + \left(R_l(1+c)u^2 + p\right)_x & = 0 \end{cases} \qquad \textbf{(P)}$$

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System **(P)** is analogous to **(H)** in Eulerian coordinates, with $\lambda = \dfrac{c}{1+c}$,
but the pressure laws differ when $\lambda\,,\, c \sim 0$.

The homogeneous system

Issue: existence of weak solutions (in the BV class) for the Cauchy problem, globally defined in time, with possibly large data:

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A strictly hyperbolic 3×3 system: $e_{1,3} = \pm\sqrt{-p_v} = \pm a(\lambda)/v, \qquad e_2 = 0$.
Two fields are GNL, one is LD .

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- For **small** BV data: well-posedness of the Cauchy problem.

[Glimm 1965; Bressan, *Hyperbolic systems...*, 2000].

Known results (partial list)

- *If λ_o is constant*: the Cauchy problem for $v_t - u_x = 0$, $u_t + (a^2/v)_x = 0$ has a global solution for every initial data (v_o, u_o) with

$$\mathrm{Tot.Var.}\,(v_o, u_o) < \infty$$

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- If $p(v) = \dfrac{a^2}{v^\gamma}$, with $\gamma > 1$ global existence if $(\gamma - 1)\mathrm{Tot.Var.}\,(u_o, v_o)$ is small [Nishida-Smoller, DiPerna 1973].

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- For the full Euler system: Liu (1977), Temple (1981).

Other known results

About p-system with $\gamma = 1$ and source term:

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- [Luo-Natalini-Yang 2000], [Amadori-Guerra, 2001]:
  global existence of BV solutions for

$$\begin{cases} v_t - u_x &= 0 \\ u_t + (1/v)_x &= \frac{1}{\tau} r(v,u)\,, \end{cases}$$

  and relaxation limit for $\tau \to 0$. Tipical case: $r(v,u) = A(v) - u\,$.

Main result for the homogeneous system

Assume: $v_o(x) \geq \underline{v} > 0$, $\lambda_o(x) \in [0,1]$ and define

$$A_o \;=\; 2 \sup \sum_{j=1}^{n} \frac{|a(\lambda(x_j)) - a(\lambda(x_{j-1}))|}{a(\lambda(x_j)) + a(\lambda(x_{j-1}))} \,.$$

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Theorem 1: For a suitable decreasing function $H : (0, 1/2] \to [0, \infty)$, if

$$A_o < \frac{1}{2}\,,$$

$$\text{Tot.Var.}\,(\log p_o) + \frac{1}{\inf a_o} \text{Tot.Var.}\, u_o < H(A_o)\,,$$

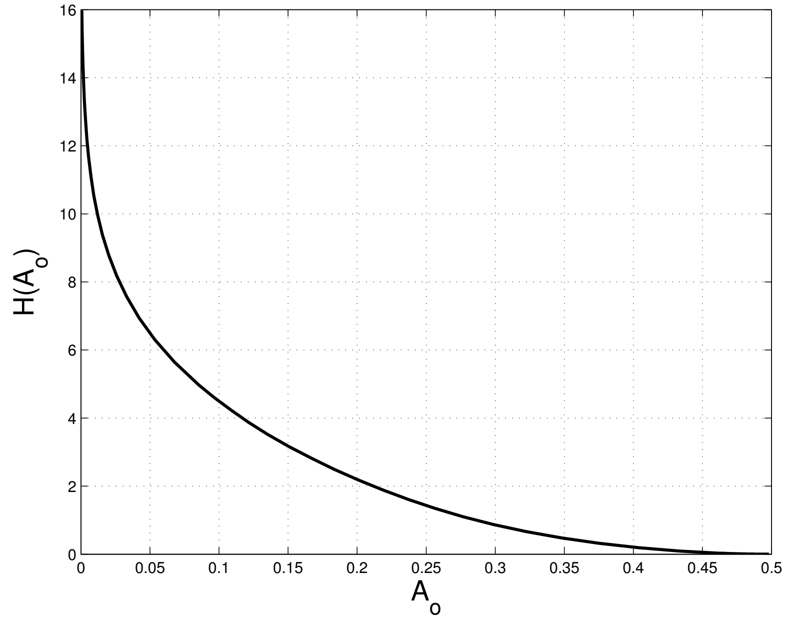
then the Cauchy problem for **(H)** has a weak entropic solution (v, u, λ) defined for $t \geq 0$, with uniformly bounded total variation.

The function H can be explicitly computed. It satisfies

$$H : (0, 1/2] \to [0, \infty)\,, \quad H(1/2) = 0\,, \quad \lim_{A \to 0+} H(A) = +\infty\,.$$

Note that: the smaller is A_o, the larger is $H(A_o)$ (recall Nishida-Smoller).

Graph of $H(A_o)$:



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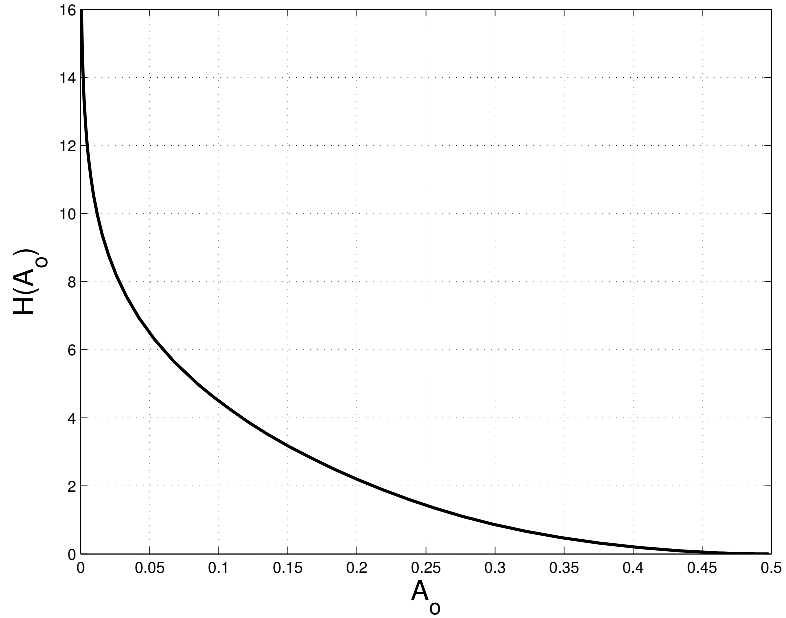

[Amadori, Corli, SIAM J. Math. Anal., 2008]

The Riemann problem

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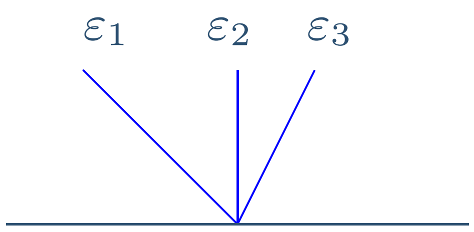

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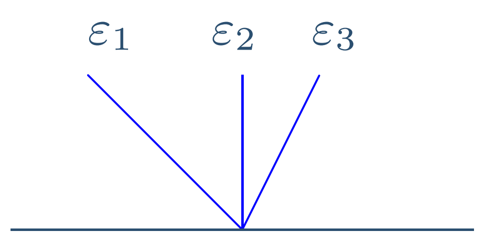

- Phase waves are stationary: p, u are conserved across them (a “kinetic condition”).

- Strength of the 1-, 3- waves:

$$|\varepsilon_{1,3}| = \frac{1}{2}\left|\log\left(\frac{v_r}{v_\ell}\right)\right| = \frac{1}{2}\left|\log\left(\frac{p_r}{p_\ell}\right)\right| .$$

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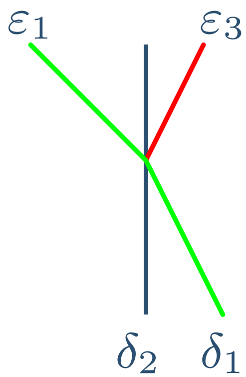
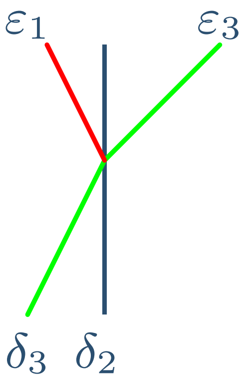
Interactions with phase waves





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$$|\varepsilon_1| + |\varepsilon_3| \leq \begin{cases} |\delta_1| + |\delta_1|[\delta_2]_+ & \text{if } 1 \text{ interacts}, \\ |\delta_3| + |\delta_3|[\delta_2]_- & \text{if } 3 \text{ interacts}. \end{cases}$$

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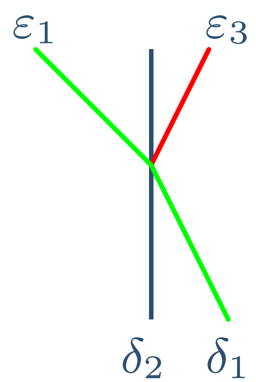

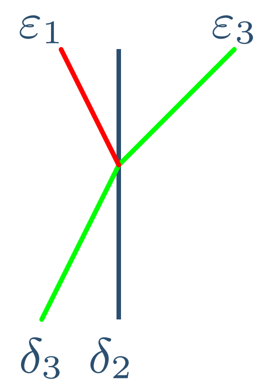

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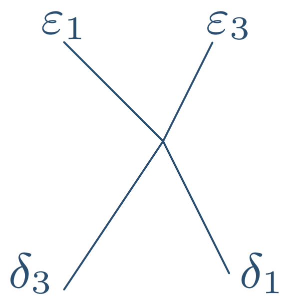
$$|\varepsilon_j| \leq \frac{1}{2}\,|\delta_2| \cdot |\delta_i| \qquad i, j = 1, 3,\ i \neq j$$

$$|\varepsilon_1| + |\varepsilon_3| \leq \begin{cases} |\delta_1| + |\delta_1|[\delta_2]_+ & \text{if } 1 \text{ interacts}, \\ |\delta_3| + |\delta_3|[\delta_2]_- & \text{if } 3 \text{ interacts}. \end{cases}$$

- The reflected wave may be very large.
- The variation $|\varepsilon_1| + |\varepsilon_3| - |\delta_i|$ may increase iff δ_i is moving toward a more liquid region.

Interactions of sonic waves – 1

- Waves of different families (1 or 3) cross each other without changing their strength:



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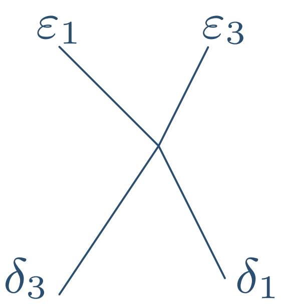

- Waves of the same family:
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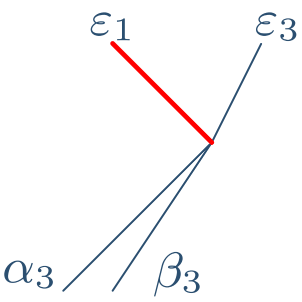

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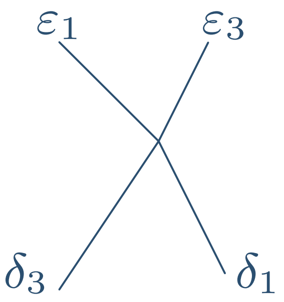

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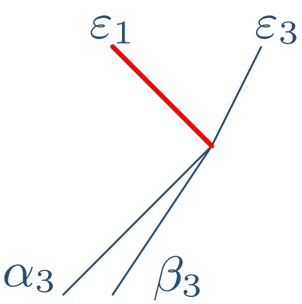

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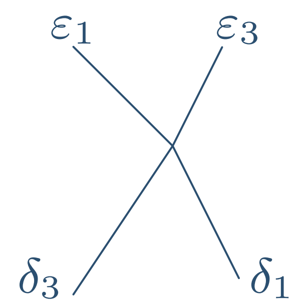

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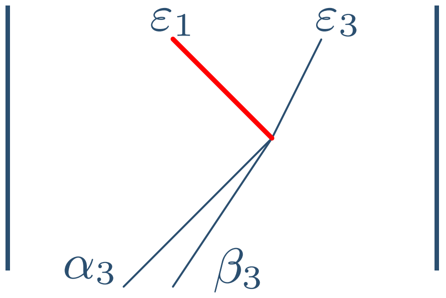


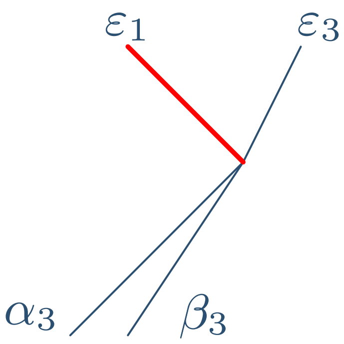
$\Rightarrow$ If λ is constant (no 2-waves are present), then $L(t) = \sum_{i=1,3} |\varepsilon_i|$ is not increasing [Nishida 68].

However, large reflected waves may interact with phase waves, producing even larger waves...

$\Rightarrow$ **Needed:** improved estimates on the reflected waves.

Interactions of sonic waves – 2

Lemma: Let two waves of the same family, of sizes α_i and β_i $(i = 1, 3)$ interact, producing ε_1, ε_3; assume that for $m > 0$



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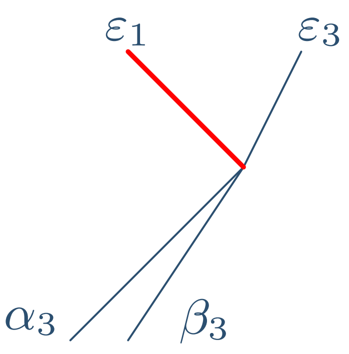

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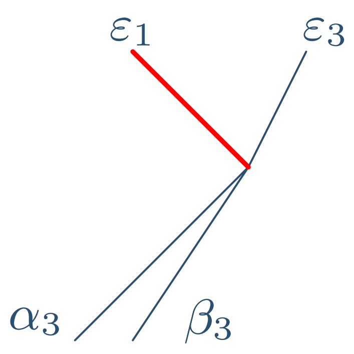

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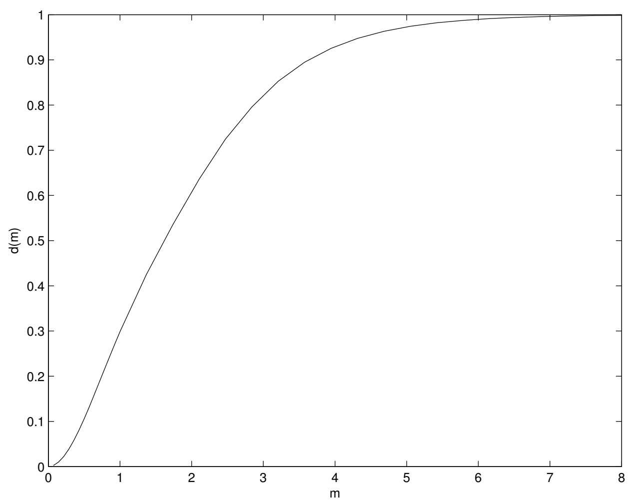

$$|\varepsilon_j| \;\leq\; d(m) \cdot \min\{|\alpha_i|, |\beta_i|\}\,, \qquad j \neq i\,.$$

Note: $d(m) \to 1$ as $m \to \infty$

The algorithm

Use a suitable version of a wave-front tracking algorithm (mainly from Bressan (2000) and Amadori-Guerra (2001)):

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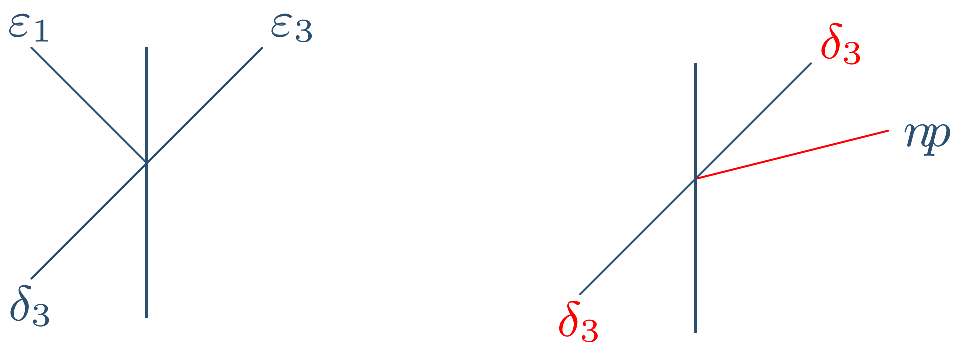

The functionals

Introduce $\xi \geq 1$, $K \geq 0$ and define

$$L_\xi = \sum_{i=1,3,\,\mathrm{rar}} |\gamma_i| + \xi \sum_{i=1,3,\,\mathrm{sh}} |\gamma_i| + K_{np} \sum_{\gamma \in \mathcal{NP}} |\gamma|$$

$$Q = \sum_{\gamma_i,\,\delta_2 \text{ approaching},\, i=1,3} |\gamma_i||\delta_2|$$

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- After a suitable choice of the coefficients ξ and K (neither too small, nor too large), one finds

$$\Delta F(t) \leq 0 \qquad \text{for all } t.$$

Convergence and consistence

- Prove that the number of interactions is finite in finite time.

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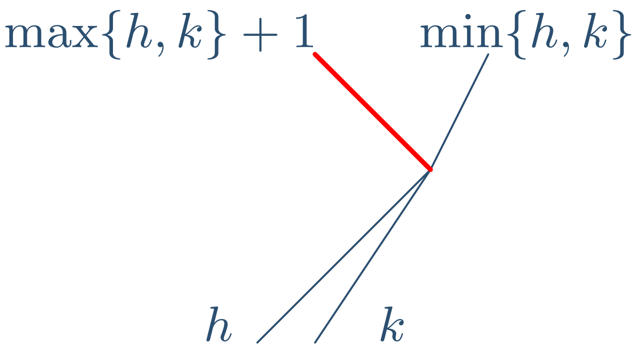

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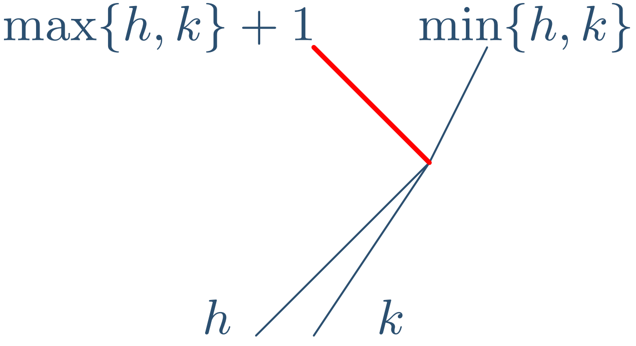

Lemma [A contraction property]
For some $0 < \mu < 1$, and for every $k \in \mathbb{N}$,

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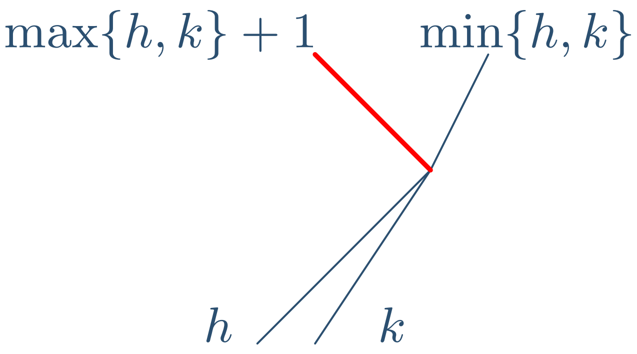

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- Pass to the limit by compactness.

The system with a reaction term

We come back to the complete system:

$$\textbf{(HS)} \qquad \begin{cases} v_t - u_x & = 0 \\ u_t + p(v,\lambda)_x & = 0 \\ \lambda_t & = \dfrac{1}{\tau} g(p,\lambda)\,, \qquad g = (p - p_e)\lambda(\lambda - 1) \end{cases}$$

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Let $E = \{(v,u,\lambda) : g = 0\}$ be the set of equilibrium points of the source term. It consists of 3 subsets:

- E_o, $\lambda = 0$, liquid phase: the system reduces to

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- E_p, equilibrium pressure. Here the system reduces to

$$p = p_e\,, \qquad u = const.\,, \qquad \lambda = \lambda(x)\,.$$

The case $\lambda \sim 0$

For $\tau > 0$ fixed, we focus on the case $\lambda \sim 0$. From the equation

$$\lambda_t = \frac{1}{\tau}(p - p_e)\lambda(\lambda - 1)\,,$$

note that the sign of $(p - p_e)$ determines the behavior of the equation for λ: provided that

$$p - p_e \geq c > 0\,, \qquad \lambda \leq \mu < 1$$

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[Dafermos& Hsiao, 1982]

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Theorem 2: For $\tau > 0$ fixed, assume that:

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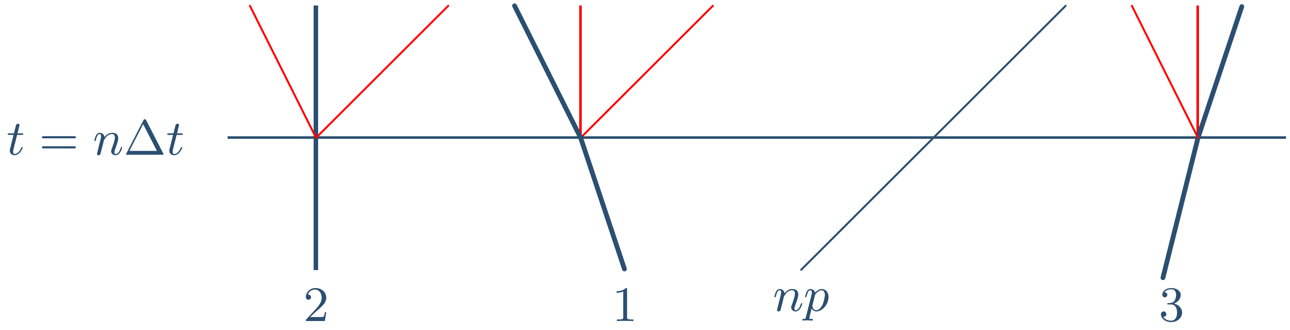
Then the Cauchy problem for the system **(HS)** has a weak entropic solution (v, u, λ) defined for $t \geq 0$, with uniformly bounded total variation.

Applying the fractional step scheme...



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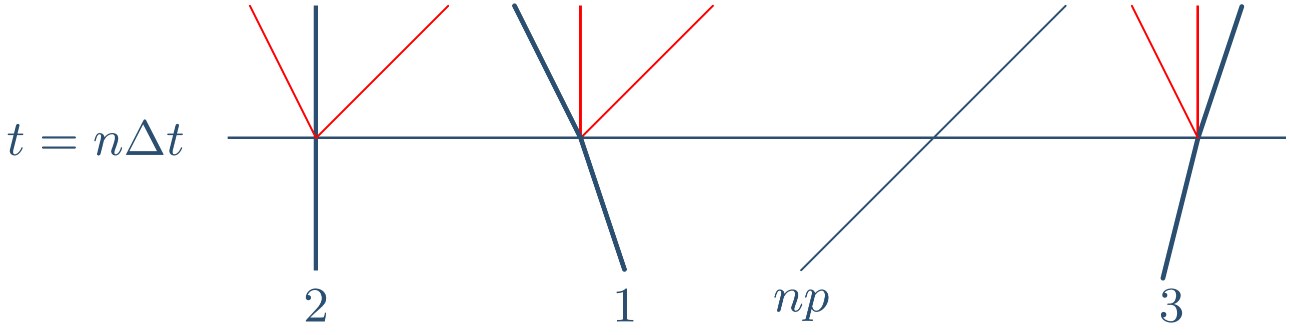

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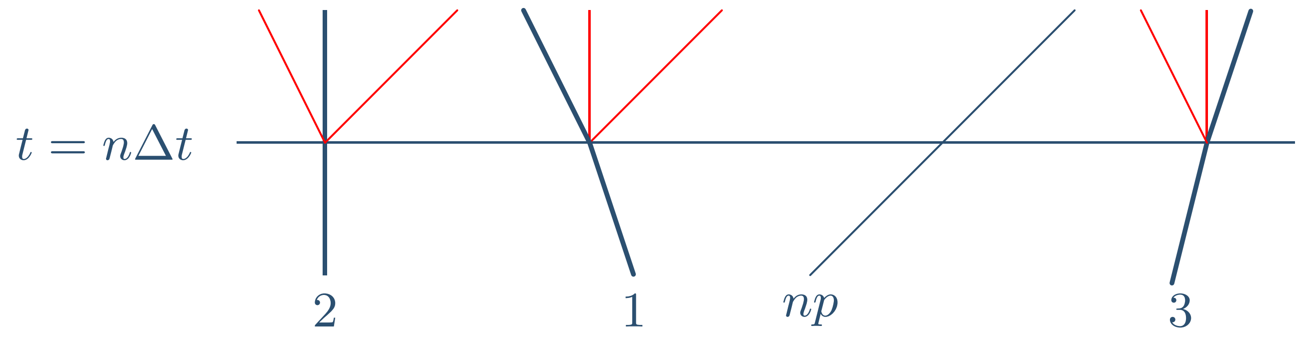

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- For every $t > s \geq 0$, one has

$$\int_a^b |\lambda(x,t) - \lambda(x,s)|\, dx \;\leq\; L_\tau\,(t - s + \Delta t)$$

$$L_\tau = C_1 \;+\; \frac{C_2}{\tau}\,\mathrm{e}^{-\frac{C_3\,s}{\tau}}\;.$$

The relaxation limit

Theorem 3: For $\tau > 0$, consider the system **(HS)** and the initial data

$$(v, u, \lambda)(0, x) = \left(v_o^\tau(x), u_o^\tau(x), \lambda_o^\tau(x)\right),$$

satisfying the bounds of Theorem 2 uniformly with respect to τ. Assume that

$$v_o^\tau \to v_o\,, \quad u_o^\tau \to u_o \qquad \text{in } L^1_{loc}(\mathbb{R})\,, \quad \text{as } \tau \to 0\,.$$

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$$\begin{array}{rcll} \lambda^{\tau_n} & \to & 0 & \text{in } L^1_{loc}(\mathbb{R} \times (0, \infty)) \\ (v^{\tau_n}, u^{\tau_n}) & \to & (\widetilde{v}, \widetilde{u}) & \text{in } L^1_{loc}(\mathbb{R} \times [0, \infty)), \end{array}$$

where $(\widetilde{v}, \widetilde{u})$ is a weak solution for $t \geq 0$ to the Cauchy problem

$$\begin{cases} v_t - u_x & = 0 \\ u_t + p(v, 0)_x & = 0, \end{cases} \qquad \begin{cases} v(x, 0) = v_o(x) \\ u(x, 0) = u_o(x). \end{cases}$$

The relaxation limit: entropies

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$$\widetilde{\eta}(v,u) = \frac{u^2}{2} - A(0)\log v, \qquad \widetilde{q}(v,u) = \frac{A(0)u}{v},$$

(entropy-entropy flux pair for the 2×2 system with $\lambda = 0$), then, for any smooth function ϕ

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- For a suitable choice of ϕ, the entropy is dissipative w.r.t. the source term. Hence one can pass to the limit $\tau_n \to 0$ and prove that the $(\widetilde{v},\widetilde{u})$ satisfies the entropy inequality for the 2×2 system w.r.t. $\widetilde{\eta}$, $\widetilde{q}$.

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  - Weak solutions

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