Multi-d shock waves and surface waves

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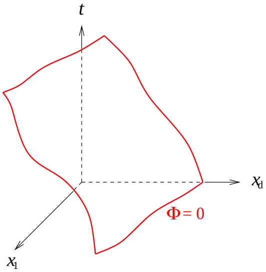
Outline

1. Multi-d shock waves stability
   - Theory
   - Examples
2. Neutral stability and well-posedness
3. Weakly nonlinear surface waves
   - Derivation of amplitude equation
   - Well-posedness for amplitude equation

General equations for a ‘shock wave’



$$\partial_t f_0(u) + \partial_j f_j(u) = 0_n\,, \qquad \Phi(t,x) \neq 0\,,$$

$$[f_0(u)]\,\partial_t \Phi + [f_j(u)]\,\partial_j \Phi = 0_n\,, \qquad \Phi(t,x) = 0\,.$$

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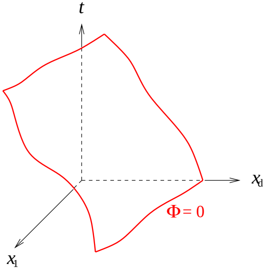

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Basic assumption: hyperbolicity in t-direction, i.e. for all $u \in \mathcal{U} \subset \mathbb{R}^n$, the matrix $A_0(u) := \mathrm{d}f_0(u)$ is nonsingular, and for all $\nu \in \mathbb{R}^d$, the matrix $A_0(u)^{-1}A_j(u)\,\nu_j$ only has real semisimple eigenvalues.

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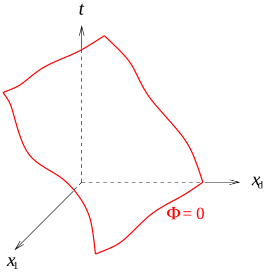

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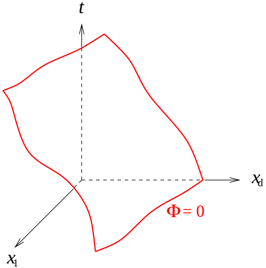

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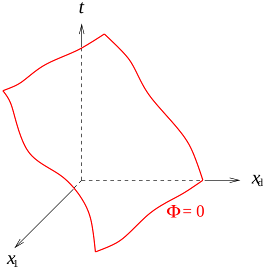
Shock is noncharacteristic iff both matrices $\mathcal{A}_0(u_\pm, \nabla\Phi)$ are nonsingular along $\Phi = 0$.

General equations for a 'shock wave'



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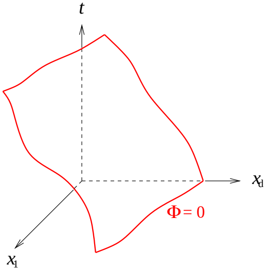
$$\boxed{\mathcal{A}_0(u,\nu) := A_0(u)^{-1}(A_0(u)\nu_0 + A_j(u)\,\nu_j)}$$

Shock is classical (or Laxian) iff
$\dim E^u(\mathcal{A}_0(u_-,\nabla\Phi)) + \dim E^s(\mathcal{A}_0(u_+,\nabla\Phi)) = n+1,$
$\dim E^u(\mathcal{A}_0(u_+,\nabla\Phi) + \dim E^s(\mathcal{A}_0(u_-,\nabla\Phi)) = n-1.$

General equations for a ‘shock wave’



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Shock is undercompressive iff

$\dim E^u(\mathcal{A}_0(u_-,\nabla\Phi)) + \dim E^s(\mathcal{A}_0(u_+,\nabla_x\Phi)) = n+1-p$,

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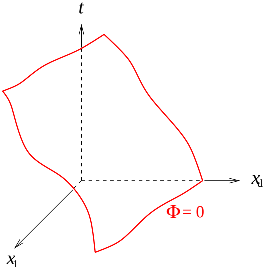
General equations for a ‘shock wave’



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$$[g_0(u)]\,\partial_t\Phi + [g_j(u)]\,\partial_j\Phi = 0_p\,, \qquad \Phi(t,x) = 0\,.$$

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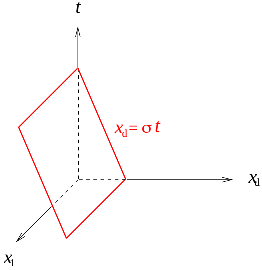
Linear analysis

Ref. planar shock



[Lopatinskiĭ'70], [Kreiss'70], [Blokhin'82], [Majda'83], [Freistühler'98].

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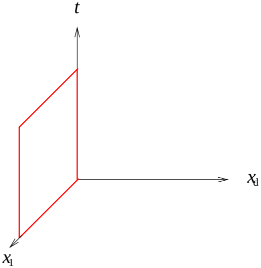

- Change frame $\Longrightarrow \sigma = 0$.

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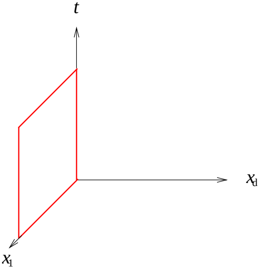

- Change frame $\Longrightarrow \sigma = 0$.
- Change coordinates $(t, x) \mapsto$
  $(t, y) := (t, x_1, \dots, x_{d-1}, \Phi(t, x))$,
  $\Phi(t, x) = x_d - \chi(t, x_1, \dots, x_{d-1})$.

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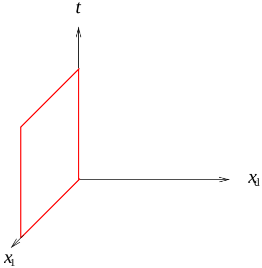

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- Linearize eqns about
  $(u, \chi) = (\underline{u}, 0)\,,\ \underline{u} := u_\pm\,,\ y_d \gtrless 0\,.$

[Lopatinskiĭ'70], [Kreiss'70], [Blokhin'82], [Majda'83], [Freistühler'98].

Normal modes analysis

Transmission problem:

$$\begin{cases} A_0(\underline{u})\partial_t u + A_j(\underline{u})\partial_j u = 0_n\,, & y_d \gtrless 0\,, \\ [F_0(\underline{u})]\,\partial_t\chi + [F_j(\underline{u})]\,\partial_j\chi \;=\; [\mathrm{d}F_d(\underline{u})\cdot u], & y_d = 0\,. \end{cases}$$

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Fourier-Laplace transform $(t,\check{y}) \rightsquigarrow (\tau,\check{\eta})$
$\Rightarrow$ shooting ODE problem.

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Normal modes:
$\chi = X\,\mathrm{e}^{\tau t + i\eta_j y_j}$, $u = U(y_d)\,\mathrm{e}^{\tau t + i\eta_j y_j}$ with $U \in L^2(\mathbb{R})$, $\mathrm{Re}\,(\tau) > 0$,
$U(0+) \in E^u(\mathcal{A}_d(u,\tau,i\check{\eta}))$ and $U(0-) \in E^s(\mathcal{A}_d(u,\tau,i\check{\eta}))$.

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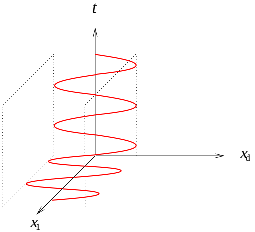
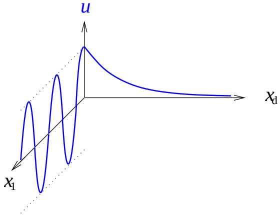
Neutral modes of finite energy, or surface waves:
$\chi = X\,\mathrm{e}^{i\eta_0 t + i\eta_j y_j}$, $u = U(y_d)\,\mathrm{e}^{i\eta_0 t + i\eta_j y_j}$ with still $U \in L^2(\mathbb{R})$,

$U(0+) \in E^u(\mathcal{A}_d(u,i\eta_0,i\check{\eta}))$ and $U(0-) \in E^s(\mathcal{A}_d(u,i\eta_0,i\check{\eta}))$.

Surface waves





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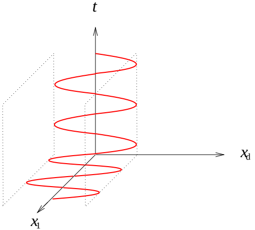
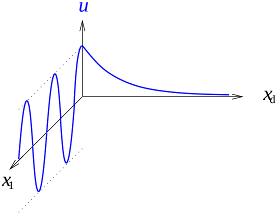


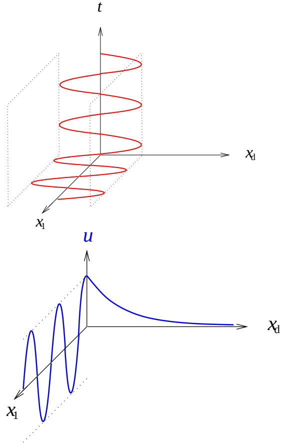


Isotropic elasticity

$$\partial_{tt} u = \lambda \Delta u + (\lambda + \mu) \nabla \mathrm{div}\, u, \quad x_2 > 0\,,$$
$$\partial_2 u_1 + \partial_1 u_2 = 0\,, \quad x_2 = 0\,,$$
$$\mu \partial_1 u_1 + (2\lambda + \mu) \partial_2 u_2 = 0\,, \quad x_2 = 0\,.$$

[Rayleigh1885] (see also [Serre'06])

Surface waves



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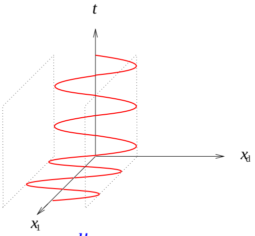
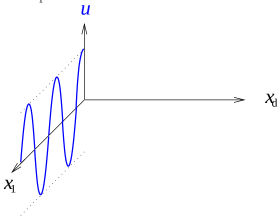
For $\lambda > 0$, $\lambda + \mu > 0$, $\exists$ Rayleigh waves, or 'Surface Acoustic Waves', of speed less than $\sqrt{\lambda}$.

[Rayleigh1885] (see also [Serre'06])

Surface waves



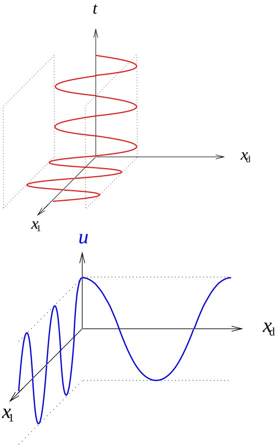


Classical shocks in gas dynamics

[Bethe'42], [D'yakov'54], [Iordanskiĭ'57], [Kontorovič'58], [Erpenbeck'62], [Majda'83], [Blokhin'82].

[Menikoff–Plohr'89], [Jenssen-Lyng'04], [SBG–Serre'07].

Surface waves



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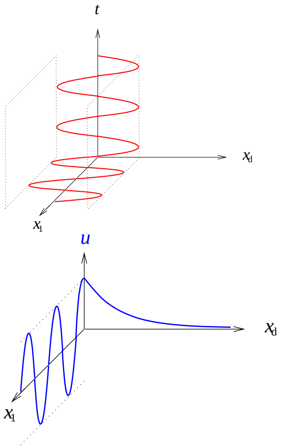
There exist neutral modes iff $1 - M < k \leq 1 + M^2(r-1)$, where M = Mach number behind the shock, $r = v_p/v_b$ with $v_{p,b}$ = volume past/behind the shock, $k = 2 + M^2 \frac{(v_b - v_p)}{T}\, p_s'$.

[Menikoff–Plohr'89], [Jenssen-Lyng'04], [SBG–Serre'07].

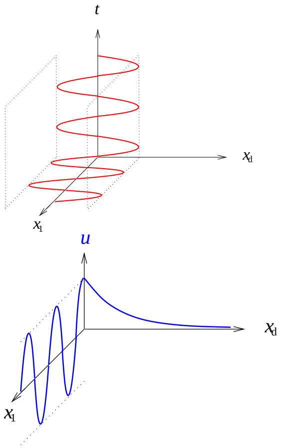
Surface waves



Phase boundaries

[SBG'98-99], [SBG–Freistühler'04]

Surface waves



Phase boundaries

[SBG'98-99], [SBG–Freistühler'04]

For nondissipative subsonic phase boundaries there exist surface waves, of speed less than $\sqrt{u_b u_p}$.

Outline

Constant coefficients linear problem

'Interior' operator $L(\underline{u}) := A_0(\underline{u})\partial_t + A_j(\underline{u})\partial_j$

'Boundary' operator $B(\underline{u}) := [F_0(\underline{u})]\,\partial_t + [F_j(\underline{u})]\,\partial_j - [\mathrm{d}F_d(\underline{u})\cdot]$

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Maximal a priori estimate

$$\gamma\,\|\mathrm{e}^{-\gamma t}u\|_{L^2}^2 + \|\mathrm{e}^{-\gamma t}u_{|y_d=0}\|_{L^2}^2 + \|\mathrm{e}^{-\gamma t}\chi\|_{H^1_\gamma}^2 \lesssim$$
$$\frac{1}{\gamma}\,\|\mathrm{e}^{-\gamma t}L(\underline{u})u\|_{L^2}^2 + \|\mathrm{e}^{-\gamma t}B(\underline{u})(\chi, u)\|_{L^2}^2$$

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$$\frac{1}{\gamma}\,\|\mathrm{e}^{-\gamma t}L(\underline{u})u\|_{L^2}^2 + \|\mathrm{e}^{-\gamma t}B(\underline{u})(\chi,u)\|_{L^2}^2$$

OK under uniform Kreiss-Lopatinskiĭ condition, i.e. without neutral modes. (Proof based on Kreiss' symmetrizers technique.)

Constant coefficients linear problem

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A priori estimate with loss of derivatives

$$\gamma\,\|\mathrm{e}^{-\gamma t}u\|^2_{L^2} + \|\mathrm{e}^{-\gamma t}u_{|y_d=0}\|^2_{L^2} + \|\mathrm{e}^{-\gamma t}\chi\|^2_{H^1_\gamma} \lesssim$$
$$\frac{1}{\gamma^3}\,\|\mathrm{e}^{-\gamma t}L(\underline{u})u\|^2_{L^2(\mathbb{R}^+;H^1_\gamma)} + \frac{1}{\gamma^2}\,\|\mathrm{e}^{-\gamma t}B(\underline{u})(\chi,u)\|^2_{H^1_\gamma}$$

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$$\frac{1}{\gamma^3}\,\|\mathrm{e}^{-\gamma t}L(\underline{u})u\|_{L^2(\mathbb{R}^+;H^1_\gamma)}^2 + \frac{1}{\gamma^2}\,\|\mathrm{e}^{-\gamma t}B(\underline{u})(\chi,u)\|_{H^1_\gamma}^2$$

Takes into account neutral modes. (Proof still based on Kreiss' symmetrizers technique [Coulombel'02], [Sablé-Tougeron'88].)

Fully nonlinear problem

Local-in-time existence of 'smooth' solutions

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- under uniform Kreiss–Lopatinskiĭ condition [Majda’83], [Blokhin’82], [Métivier *et al.*’90-00],

Fully nonlinear problem

Local-in-time existence of 'smooth' solutions

- under uniform Kreiss–Lopatinskiĭ condition [Majda'83], [Blokhin'82], [Métivier *et al.*'90-00],
- under mere Kreiss–Lopatinskiĭ condition [Coulombel–Secchi'08]: with neutral modes and characteristic modes ; application to subsonic phase boundaries and compressible 2d-vortex sheets. (Proof using Nash–Moser iteration scheme.)

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   - Examples
2. Neutral stability and well-posedness
3. Weakly nonlinear surface waves
   - Derivation of amplitude equation
   - Well-posedness for amplitude equation

Fully nonlinear problem

$$\begin{cases} A_0(u)\partial_t u + A_j(u)\partial_j u + A^d(u,\nabla\chi)\,\partial_d u = 0_n\,, & y_d \neq 0\,, \\ [F_0(u)]\,\partial_t\chi + [F_j(u)]\,\partial_j\chi = [F_d(u)]\,, & y_d = 0\,. \end{cases}$$

$$\boxed{A^d(u,\nabla\chi) := A_d(u) - A_0(u)\partial_t\chi - A_j(u)\partial_j\chi}$$

Fully nonlinear problem

$$\begin{cases} A_0(u)\partial_t u + A_j(u)\partial_j u + A^d(u,\nabla\chi)\,\partial_d u = 0_n\,, & y_d \neq 0\,, \\ J\nabla\chi + h(u) = 0_{n+p}\,, & y_d = 0\,. \end{cases}$$

$$\boxed{A^d(u,\nabla\chi) := A_d(u) - A_0(u)\partial_t\chi - A_j(u)\partial_j\chi}$$

$$J = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Fully nonlinear problem

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Asymptotic expansion

$$u = \underline{u} + \varepsilon \dot{u}(\eta_0 t + \eta_j y_j, y_d, \varepsilon t) + \varepsilon^2 \ddot{u}(\eta_0 t + \eta_j y_j, y_d, \varepsilon t) + h.o.t.$$

$$\chi = \varepsilon \dot{\chi}(\eta_0 t + \eta_j y_j, y_d, \varepsilon t) + \varepsilon^2 \ddot{\chi}(\eta_0 t + \eta_j y_j, \varepsilon t) + h.o.t.$$

[SBG-Rosini'08], [Hunter'89], [Parker'88].

Approximate problems

$\xi := \eta_0 t + \eta_j y_j,\ z := y_d,\ \tau := \varepsilon t.$

First order

$$\begin{cases} (\eta_0 A_0(\underline{u}) + \eta_j A_j(\underline{u}))\, \partial_\xi \dot{u} + A_d(\underline{u})\, \partial_z \dot{u} = 0_n\,, & z \neq 0\,, \\ J\eta\, \partial_\xi \dot{\chi} + \mathrm{d}h(\underline{u}) \cdot \dot{u} = 0_{n+p}\,, & z = 0\,, \end{cases}$$

Approximate problems

$$\xi := \eta_0 t + \eta_j y_j, \; z := y_d, \; \tau := \varepsilon t.$$

First order

$$\begin{cases} (\eta_0 A_0(\underline{u}) + \eta_j A_j(\underline{u}))\, \partial_\xi \dot{u} + A_d(\underline{u})\, \partial_z \dot{u} = 0_n \,, & z \neq 0 \,, \\ J\eta\, \partial_\xi \dot{\chi} + \mathrm{d}h(\underline{u}) \cdot \dot{u} = 0_{n+p} \,, & z = 0 \,, \end{cases}$$

Second order

$$\begin{cases} (\eta_0 A_0(\underline{u}) + \eta_j A_j(\underline{u}))\, \partial_\xi \ddot{u} + A_d(\underline{u})\, \partial_z \ddot{u} = \dot{M} \,, & z \neq 0 \,, \\ J\eta\, \partial_\xi \ddot{\chi} + \mathrm{d}h(\underline{u}) \cdot \ddot{u} = \dot{G} \,, & z = 0 \,, \end{cases}$$

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$$\begin{aligned} -\dot{M} := \ & A_0(\underline{u})\partial_\tau \dot{u} + (\eta_0 \mathrm{d}A_0(\underline{u}) + \eta_j \mathrm{d}A_j(\underline{u})) \cdot \dot{u} \cdot \partial_\xi \dot{u} \\ & + \mathrm{d}A_d(\underline{u}) \cdot \dot{u} \cdot \partial_z \dot{u} - (\partial_\xi \dot{\chi})(\eta_0 A_0(\underline{u}) + \eta_j A_j(\underline{u}))\partial_z \dot{u} \end{aligned}$$

Approximate problems

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$$-\dot{G} := (\partial_\tau \dot{\chi}) e_1 \;+\; \tfrac{1}{2}\, \mathrm{d}^2 h(\underline{u}) \cdot (\dot{u}, \dot{u}) \,.$$

Transformation of approximate problems

- Fourier transform $\xi \rightsquigarrow k$,

First order

$$\begin{cases} ik\,(\eta_0 A_0(\underline{u}) + \eta_j A_j(\underline{u}))\,\dot{u} + A_d(\underline{u})\,\partial_z \dot{u} = 0_n\,, & z \neq 0\,, \\ ik\,J\eta\,\dot{\chi} + \mathrm{d}h(\underline{u}) \cdot \dot{u} = 0_{n+p}\,, & z = 0\,, \end{cases}$$

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$$\begin{cases} \mathcal{L}(\underline{u}; k\eta) \cdot \dot{u} = 0_n \,, & z \neq 0 \,, \\ C(\underline{u}; \eta)\dot{u} = 0_{n+p-1} \,, & z = 0 \,, \end{cases}$$

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$$\mathcal{L}(\underline{u}; \eta) := i\eta_0 A_0(\underline{u}) + i\eta_j A_j(\underline{u}) + A_d(\underline{u})\,\partial_z \,.$$

Resolution of approximate problems

First order level

- Existence of linear surface wave $\Longrightarrow L^2(\mathrm{d}z)$ solution $\dot{u}_1$ of

$$\begin{cases} \mathcal{L}(\underline{u};\eta)\cdot\dot{u}_1 = 0_n\,, & z \neq 0\,,\\ C(\underline{u};\eta)\dot{u}_1 = 0_{n+p-1} & z = 0\,. \end{cases}$$

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- Homogeneity $\Longrightarrow$ other square integrable solutions of first order system of the form $\dot{u}(k,z,\tau) = W(k,\tau)\,\dot{u}_1(kz)$, $k>0$.

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Second order level

- Solvability condition by means of an adjoint problem.

Solvability of second order problem

$$(\cdot\cdot) \quad \begin{cases} \mathcal{L}(\underline{u}; k\eta) \cdot \ddot{u} = \dot{M}\,, & z \neq 0\,, \\ C(\underline{u};\eta)\ddot{u} = T(\eta)\dot{G}\,, & z = 0\,, \end{cases}$$

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- $\begin{pmatrix} -A_d(u-) & 0 \\ 0 & A_d(u_+) \end{pmatrix} = \begin{pmatrix} -D_-^* \\ D_+^* \end{pmatrix} N + P^*\,(-C_-|C_+)\,.$

Solvability of second order problem

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- $\begin{pmatrix} -A_d(u-) & 0 \\ 0 & A_d(u_+) \end{pmatrix} = \begin{pmatrix} -D_-^* \\ D_+^* \end{pmatrix} N + P^* \left(-C_- | C_+\right)\,.$
- There exists a $L^2(\mathrm{d}z)$ solution $\ddot{u}$ of $(\cdot\cdot)$ iff

$$\boxed{\int v^* \dot{M}\,\mathrm{d}z \;+\; (v(0-)^* | v(0+)^*) P T \dot{G} \;=\; 0\,,}$$

with v solution of $\begin{cases} \mathcal{L}(\underline{u}; k\eta)^* \cdot v = 0_n\,, & z \neq 0\,, \\ D_+ v(0+) - D_- v(0-) = 0_{n-p+1}\,, & z = 0\,. \end{cases}$

Resulting amplitude equation

Nonlocal generalisation of Burgers' equation:

$$\partial_\tau w + \partial_\xi Q[w] = 0\,,$$

$$\mathscr{F}(Q[w])(k) = \int_{-\infty}^{+\infty} \Lambda(k-\ell,\ell)\widehat{w}(k-\ell)\widehat{w}(\ell)\mathrm{d}\ell\,.$$

with piecewise smooth kernel Λ, homogeneous degree 0.

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with piecewise smooth kernel Λ, homogeneous degree 0.

- Recover classical inviscid Burgers equation if $\Lambda \equiv 1/2$ (arises in case of neutral modes of infinite energy [Artola-Majda'87]).

Nonlocal Burgers equations

In Fourier variables:

$$\partial_\tau \widehat{w} + ik \int_{-\infty}^{+\infty} \Lambda(k-\ell,\ell)\widehat{w}(k-\ell,\tau)\widehat{w}(\ell,\tau)\mathrm{d}\ell = 0\,.$$

Existence of smooth solutions? Well-posedness?

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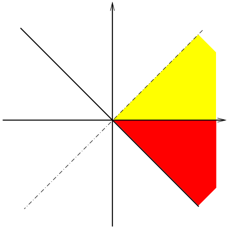

Properties of Λ:

Nonlocal Burgers equations

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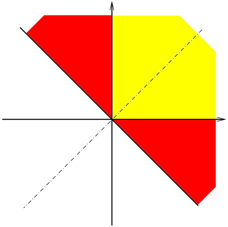

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- $\Lambda(k,\ell) = \Lambda(\ell,k)$

Nonlocal Burgers equations

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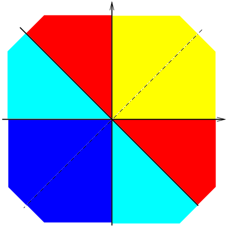

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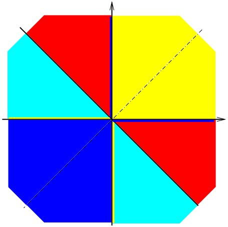

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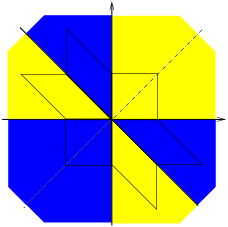

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Hamiltonian nonlocal Burgers equations

$$\left.\begin{aligned} \Lambda(k,\ell) &= \Lambda(\ell,k) \\ \Lambda(-k,-\ell) &= \overline{\Lambda(k,\ell)} \\ \Lambda(k+\xi,-\xi) &= \overline{\Lambda(k,\xi)} \end{aligned}\right\} \Longrightarrow \text{Hamiltonian structure :}$$

$$\boxed{\partial_\tau w + \partial_x \delta\mathcal{H}[w] = 0\,,}$$

$$\mathcal{H}[w] := \frac{1}{3} \iint \Lambda(k-\ell,\ell)\widehat{w}(k-\ell)\widehat{w}(\ell)\widehat{w}(-k)\,\mathrm{d}k\,\mathrm{d}\ell\,.$$

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$\implies$ Local existence of smooth periodic solutions [Hunter'06] (also see [Alì–Hunter–Parker'02]).

Stable nonlocal Burgers equations

$$\left.\begin{aligned} \Lambda(k,\ell) &= \Lambda(\ell,k) \\ \Lambda(-k,-\ell) &= \overline{\Lambda(k,\ell)} \\ \Lambda(1,0-) &= \overline{\Lambda(1,0+)} \end{aligned}\right\} \Longrightarrow \text{a priori estimates}\,,$$

Stable nonlocal Burgers equations

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and eventually local H^2 well-posedness [SBG'08].

A priori estimates

- Local Burgers:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int(\partial_\xi^n w)^2 \lesssim \|\partial_\xi w\|_{L^\infty}\int(\partial_\xi^n w)^2 .$$

A priori estimates

- Local Burgers:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int(\partial_\xi^n w)^2 \lesssim \|\partial_\xi w\|_{L^\infty}\int(\partial_\xi^n w)^2\,.$$

- Nonlocal Burgers:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int(\partial_\xi^n w)^2 \lesssim \|\mathscr{F}(\partial_\xi w)\|_{L^1}\int(\partial_\xi^n w)^2\,.$$

A priori estimates

L^2 estimate $(n = 0)$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int w^2 \mathrm{d}\xi = \frac{\mathrm{d}}{\mathrm{d}\tau}\int |\widehat{w}|^2 \mathrm{d}k =$$

$$-2\,\mathrm{Re}\left(\iint i\,k\,\Lambda(k-\ell,\ell)\widehat{w}(k-\ell)\widehat{w}(\ell)\widehat{w}(-k)\,\mathrm{d}\ell\mathrm{d}k\right)$$

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$$\leq 2\,\|\Lambda\|_{L^\infty}\,\|\widehat{w}\|_{L^2}^2 \int |k\widehat{w}(k)|\,\mathrm{d}k$$

by Fubini and Cauchy-Schwarz!

A priori estimates

H^1 estimate $(n = 1)$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int (\partial_\xi w)^2 \mathrm{d}\xi \; = \; \frac{\mathrm{d}}{\mathrm{d}\tau} \int k^2 \, |\widehat{w}|^2 \mathrm{d}k \; =$$

$$-2\,\mathrm{Re}\left(\iint i\, k^3 \, \Lambda(k-\ell,\ell) \widehat{w}(k-\ell) \widehat{w}(\ell) \widehat{w}(-k) \, \mathrm{d}\ell \mathrm{d}k \right)$$

A priori estimates

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$$\left|\iint_{|k|\leq|\ell|}\ldots\right| \leq \|\Lambda\|_{L^\infty}\,\|\widehat{\partial_\xi w}\|_{L^1}\,\|\widehat{\partial_\xi w}\|_{L^2}^2\,.$$

A priori estimates

H^1 estimate (cont.)

$$\mathrm{Re}\left(\iint_{|k|>|\ell|} i\,k^2\,(k-\ell)\,\Lambda(k-\ell,\ell)\widehat{w}(k-\ell)\widehat{w}(\ell)\widehat{w}(-k)\,\mathrm{d}\ell\mathrm{d}k\right) =$$

$$i\iint_{|k|>|\ell|} k^2(k-\ell)\,\Lambda(k-\ell,\ell)\widehat{w}(k-\ell)\widehat{w}(\ell)\widehat{w}(-k)\,\mathrm{d}\ell\mathrm{d}k$$
$$-\quad i\iint_{|k|>|\ell|} k^2(k-\ell)\,\Lambda(\ell-k,-\ell)\widehat{w}(\ell-k)\widehat{w}(-\ell)\widehat{w}(k)\,\mathrm{d}\ell\mathrm{d}k\,.$$

A priori estimates

H^1 estimate (cont.)

$$\mathrm{Re}\left(\iint_{|k|>|\ell|} i\,k^2\,(k-\ell)\,\Lambda(k-\ell,\ell)\widehat{w}(k-\ell)\widehat{w}(\ell)\widehat{w}(-k)\,\mathrm{d}\ell\mathrm{d}k\right) =$$

$$i\iint_{|k|>|\ell|} k(k-\ell)\,\big((k-\ell)\,\Lambda(\ell-k,-\ell)-k\Lambda(k,-\ell)\big)\times \widehat{w}(\ell-k)\widehat{w}(-\ell)\widehat{w}(k)\,\mathrm{d}\ell\mathrm{d}k\,.$$

after change of variables $(k,\ell)\mapsto(k-\ell,-\ell)$ in first integral.

Local well-posedness

Theorem ([SBG'08])

If Λ is smooth outside the lines $k = 0$, $\ell = 0$, and $k + \ell = 0$, homogeneous degree zero, preserves real-valued functions, and satisfies the stability condition $\Lambda(1, 0-) = \Lambda(-1, 0-)$, then for all $w_0 \in H^2(\mathbb{R})$ there exists $T > 0$ and a unique solution $w \in \mathscr{C}(0, T; H^2(\mathbb{R})) \cap \mathscr{C}^1(0, T; H^1(\mathbb{R}))$ such that $w(0) = w_0$ of the nonlocal Burgers equation of kernel Λ, and the mapping

$$\begin{array}{ccc} H^2(\mathbb{R}) & \to & \mathscr{C}(0, T; H^2(\mathbb{R})) \\ w_0 & \mapsto & w \end{array}$$

is continuous.

Blow-up criterion

The solution w can be extended beyond T provided that $\int_0^T \|\mathscr{F}(\partial_\xi w)\|_{L^1}$ is finite.

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- elasticity: $\Lambda(k+\xi,-\xi) = \overline{\Lambda(k,\xi)}$,

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Applications

- elasticity: $\Lambda(k+\xi,-\xi) = \overline{\Lambda(k,\xi)}$,
- phase boundaries: $\Lambda(1,0-) \neq \overline{\Lambda(1,0+)}$!