Applications of dispersive estimates to the acoustic pressure waves for incompressible fluid problems

Donatella Donatelli

donatell@univaq.it

Dipartimento di Matematica Pura ed Applicata
Università di L'Aquila
67100 L'Aquila, Italy

What is an acoustic pressure wave?

$$\rho_t + (\rho u)_x = 0$$

$$\rho u_t + \rho u u_x + p_x = 0$$

What is an acoustic pressure wave?

$$\rho_t + (\rho u)_x = 0$$

$$\rho u_t + \rho u u_x + p_x = 0$$

equilibrium state: $\rho = \rho_0$, $p = p(\rho_0) = p_0$

What is an acoustic pressure wave?

$$\rho_t + (\rho u)_x = 0$$

$$\rho u_t + \rho u u_x + p_x = 0$$

equilibrium state: $\rho = \rho_0$, $p = p(\rho_0) = p_0$
for small perturbations:

$$p - p_0 = c^2(\rho - \rho_0), \; c = \sqrt{p'(\rho_0)} = \text{sound speed}$$

What is an acoustic pressure wave?

$$\rho_t + (\rho u)_x = 0$$
$$\rho u_t + \rho u u_x + p_x = 0$$

equilibrium state: $\rho = \rho_0$, $p = p(\rho_0) = p_0$
for small perturbations:

$$p - p_0 = c^2(\rho - \rho_0), \; c = \sqrt{p'(\rho_0)} = \text{sound speed}$$

linearizing the previous equations

$$\rho_t + \rho_0 u_x = 0$$
$$\rho_0 u_t + p_x = 0$$

What is an acoustic pressure wave?

$$\rho_t + (\rho u)_x = 0$$
$$\rho u_t + \rho u u_x + p_x = 0$$

equilibrium state: $\rho = \rho_0$, $p = p(\rho_0) = p_0$
for small perturbations:

$$p - p_0 = c^2(\rho - \rho_0), \; c = \sqrt{p'(\rho_0)} = \text{sound speed}$$

linearizing the previous equations

$$\frac{1}{c^2} p_t + \rho_0 u_x = 0$$
$$\rho_0 u_t + p_x = 0$$

What is an acoustic pressure wave?

$$\rho_t + (\rho u)_x = 0$$

$$\rho u_t + \rho u u_x + p_x = 0$$

equilibrium state: $\rho = \rho_0$, $p = p(\rho_0) = p_0$
for small perturbations:

$$p - p_0 = c^2(\rho - \rho_0), \; c = \sqrt{p'(\rho_0)} = \text{sound speed}$$

linearizing the previous equations

$$\frac{1}{c^2} p_t + \rho_0 u_x = 0$$

$$\rho_0 u_t + p_x = 0$$

acoustic pressure wave

$$p_{tt} - c^2 p_{xx} = 0$$

Mach number and acoustic waves

$$p_{tt} - c^2 p_{xx} = 0$$

Mach number and acoustic waves

$$p_{tt} - c^2 p_{xx} = 0$$

Mach number$= \dfrac{|u_m|}{c} = M$ $\qquad |u_m| =$ typical fluid speed

Mach number and acoustic waves

$$p_{tt} - c^2 p_{xx} = 0$$

Mach number$= \dfrac{|u_m|}{c} = M \qquad |u_m| =$ typical fluid speed

the acoustic wave contains both low and fast speed of propagation

$$p_{tt} - \frac{|u_m|^2}{M^2} p_{xx} = 0$$

Mach number and acoustic waves

$$p_{tt} - c^2 p_{xx} = 0$$

Mach number$=\dfrac{|u_m|}{c} = M$ $\qquad |u_m| =$ typical fluid speed

the acoustic wave contains both low and fast speed of propagation

$$p_{tt} - \frac{|u_m|^2}{M^2} p_{xx} = 0$$

$M \longrightarrow 0$?

Mach number and acoustic waves

$$p_{tt} - c^2 p_{xx} = 0$$

Mach number$= \frac{|u_m|}{c} = M$ $\qquad |u_m| =$ typical fluid speed

the acoustic wave contains both low and fast speed of propagation

$$p_{tt} - \frac{|u_m|^2}{M^2} p_{xx} = 0$$

$M \to 0 \Longrightarrow$ fast pressure wave speed $\Longrightarrow$

Mach number and acoustic waves

$$p_{tt} - c^2 p_{xx} = 0$$

Mach number$= \dfrac{|u_m|}{c} = M$ $\qquad |u_m| =$ typical fluid speed

the acoustic wave contains both low and fast speed of propagation

$$p_{tt} - \frac{|u_m|^2}{M^2} p_{xx} = 0$$

$M \to 0 \Longrightarrow$ fast pressure wave speed $\Longrightarrow$ fast pressure equalization

$\Longrightarrow$

Mach number and acoustic waves

$$p_{tt} - c^2 p_{xx} = 0$$

Mach number$= \dfrac{|u_m|}{c} = M$ $\qquad |u_m| =$ typical fluid speed

the acoustic wave contains both low and fast speed of propagation

$$p_{tt} - \frac{|u_m|^2}{M^2} p_{xx} = 0$$

$M \to 0 \implies$ fast pressure wave speed $\implies$ fast pressure equalization $\implies$ the pressure becomes nearly constant $\implies$

Mach number and acoustic waves

$$p_{tt} - c^2 p_{xx} = 0$$

Mach number$= \dfrac{|u_m|}{c} = M$ $|u_m| =$ typical fluid speed

the acoustic wave contains both low and fast speed of propagation

$$p_{tt} - \frac{|u_m|^2}{M^2} p_{xx} = 0$$

$M \to 0 \Longrightarrow$ fast pressure wave speed $\Longrightarrow$ fast pressure equalization $\Longrightarrow$ the pressure becomes nearly constant $\Longrightarrow$ the fluid cannot generate density variations $\Longrightarrow$

Mach number and acoustic waves

$$p_{tt} - c^2 p_{xx} = 0$$

Mach number$= \frac{|u_m|}{c} = M \qquad |u_m| =$ typical fluid speed

the acoustic wave contains both low and fast speed of propagation

$$p_{tt} - \frac{|u_m|^2}{M^2} p_{xx} = 0$$

$M \to 0 \implies$ fast pressure wave speed $\implies$ fast pressure equalization $\implies$ the pressure becomes nearly constant $\implies$ the fluid cannot generate density variations $\implies$ incompressible fluid

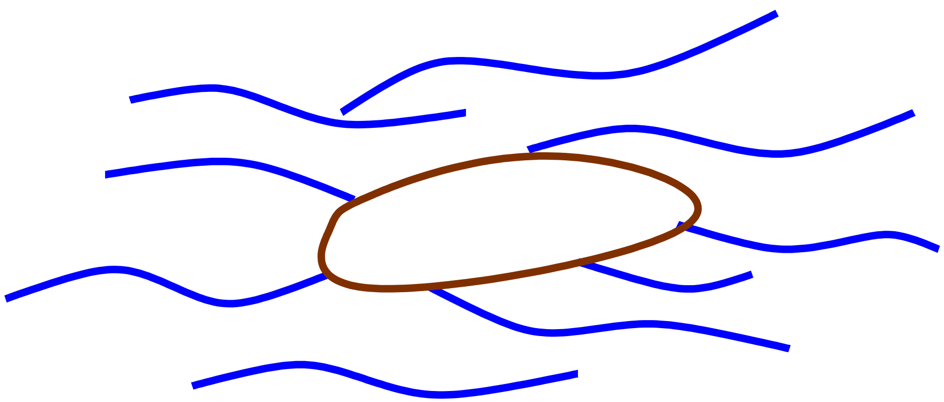
Navier Stokes equations in an exterior domain

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = f, \\ \operatorname{div} u = 0, \\ u(0, \cdot) = u_0(\cdot), \quad x \in \Omega, \quad t \geq 0 \\ u(x, t) = 0, \quad x \in \partial\Omega, \quad t \geq 0 \end{cases}$$

Navier Stokes equations in an exterior domain

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = f, \\ \operatorname{div} u = 0, \\ u(0, \cdot) = u_0(\cdot), \quad x \in \Omega, \quad t \geq 0 \\ u(x, t) = 0, \quad x \in \partial\Omega, \quad t \geq 0 \end{cases}$$

fluid flow outside a convex compact obstacle K



$\Omega = \mathbb{R}^3 - K$

Motivations

river flow around stones

Motivations

bubbles in ocean

Motivations

rain drops falling within clouds

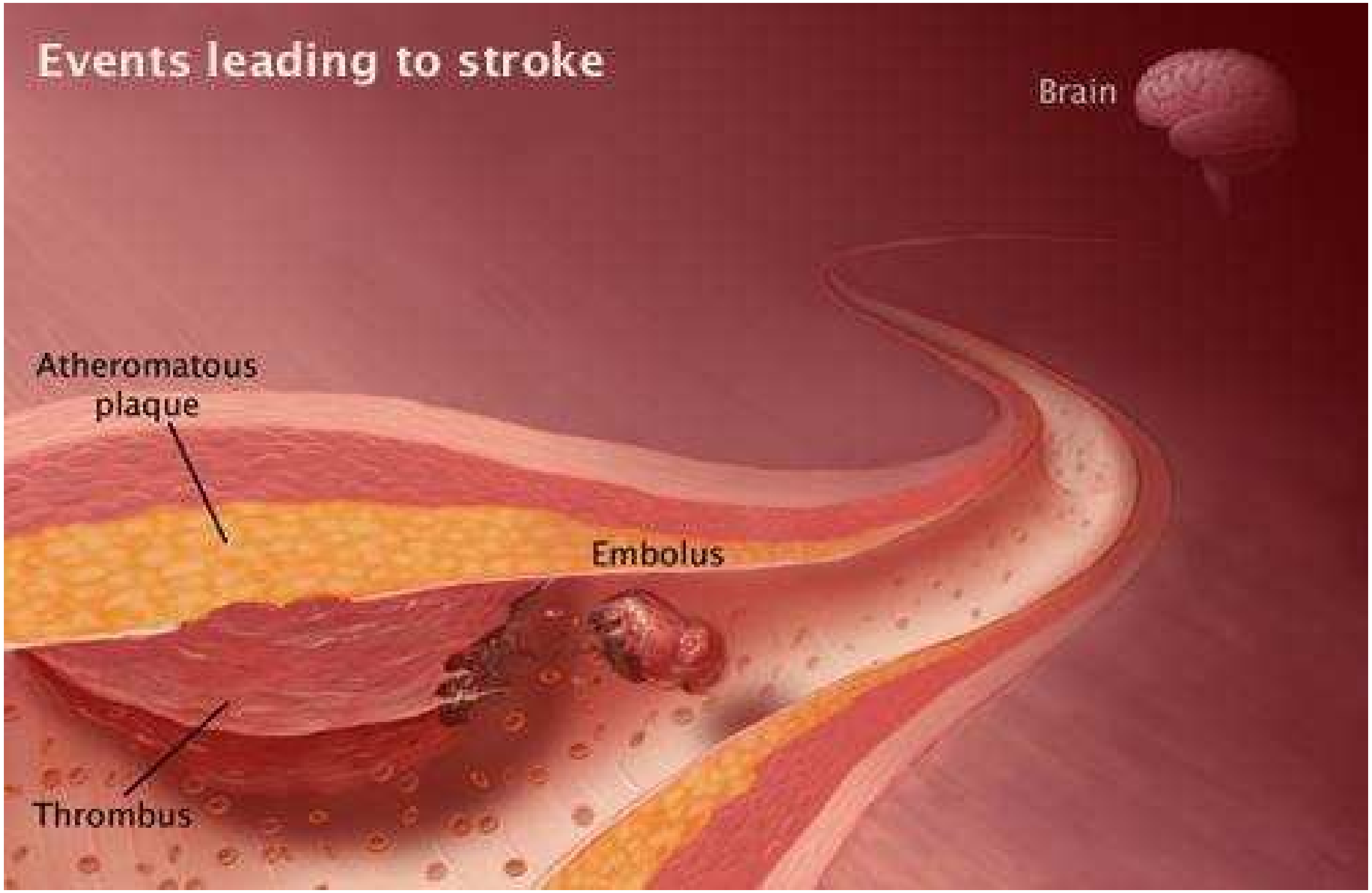
Motivations

modeling of aircrafts

Motivations

space mission

Motivations



blood flow around embolus

Existence Leray '34

- u satisfies the NS equation in the sense of distribution

$$\int_0^T \int_\Omega \left(\nabla u \cdot \nabla \varphi - u_i u_j \partial_i \varphi_j - u \cdot \frac{\partial \varphi}{\partial t} \right) dxdt$$
$$= \int_0^T \langle f, \varphi \rangle_{H^{-1} \times H_0^1} dxdt + \int_\Omega u_0 \cdot \varphi dx,$$

for all $\varphi \in C_0^\infty(\Omega \times [0,T])$, $\operatorname{div} \varphi = 0$ and $\operatorname{div} u = 0$ in $\mathcal{D}'(\Omega \times [0,T])$

Existence Leray '34

- u satisfies the NS equation in the sense of distribution

$$\int_0^T \int_\Omega \left(\nabla u \cdot \nabla \varphi - u_i u_j \partial_i \varphi_j - u \cdot \frac{\partial \varphi}{\partial t} \right) dxdt$$
$$= \int_0^T \langle f, \varphi \rangle_{H^{-1} \times H_0^1} dxdt + \int_\Omega u_0 \cdot \varphi dx,$$

for all $\varphi \in C_0^\infty(\Omega \times [0,T])$, $\operatorname{div} \varphi = 0$ and $\operatorname{div} u = 0$ in $\mathcal{D}'(\Omega \times [0,T])$

- the following energy inequality hold

$$\frac{1}{2} \int_\Omega |u(x,t)|^2 dx + \mu \int_0^t \int_\Omega |\nabla u(x,t)|^2 dxds$$
$$\leq \frac{1}{2} \int_\Omega |u_0|^2 dx + \int_0^t \langle f, u \rangle_{H^{-1} \times H_0^1} ds, \qquad \text{for all } t \geq 0.$$

Artificial compressibility in Ω

$$\begin{cases} \partial_t u^{\varepsilon} + \nabla p^{\varepsilon} = \mu \Delta u^{\varepsilon} - (u^{\varepsilon} \cdot \nabla)\, u^{\varepsilon} - \dfrac{1}{2}(\operatorname{div} u^{\varepsilon}) u^{\varepsilon} \\ \varepsilon \partial_t p^{\varepsilon} + \operatorname{div} u^{\varepsilon} = 0 \end{cases}$$

Artificial compressibility in Ω

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \dfrac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \end{cases}$$

$$\varepsilon \to 0$$

Artificial compressibility in Ω

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \frac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \end{cases}$$

$$\varepsilon \to 0$$

$$\begin{cases} \partial_t u + \nabla p = \mu \Delta u - (u \cdot \nabla)\, u \\ \operatorname{div} u = 0 \end{cases}$$

Artificial compressibility in Ω

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \frac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \end{cases}$$

non increasing kinetic energy constraint

$$\varepsilon \to 0$$

$$\begin{cases} \partial_t u + \nabla p = \mu \Delta u - (u \cdot \nabla)\, u \\ \operatorname{div} u = 0 \end{cases}$$

Artificial compressibility in Ω

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \dfrac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \end{cases}$$

“linearized” compressibility constraint

$$\varepsilon \to 0$$

$$\begin{cases} \partial_t u + \nabla p = \mu \Delta u - (u \cdot \nabla)\, u \\ \operatorname{div} u = 0 \end{cases}$$

Artificial compressibility in Ω

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \dfrac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \end{cases}$$

$$\varepsilon \to 0$$

$$\begin{cases} \partial_t u + \nabla p = \mu \Delta u - (u \cdot \nabla)\, u \\ \operatorname{div} u = 0 \end{cases}$$

The hyperbolicity of the approximation provides dispersive estimates

The convergence will be obtained via dispersion and not via compactness

Artificial compressibility approximation

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \dfrac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \end{cases}$$

Artificial compressibility approximation

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \frac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \end{cases}$$

Initial conditions:

$$u^\varepsilon(x,0) = u_0^\varepsilon(x), \qquad p^\varepsilon(x,0) = p_0^\varepsilon(x),$$

"initial layer" phenomenon for the pressure initial datum

Artificial compressibility approximation

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \dfrac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \end{cases}$$

Initial conditions:

$$u_0^\varepsilon = u^\varepsilon(\cdot, 0) \longrightarrow u_0 = u(\cdot, 0) \text{ strongly in } L^2(\Omega)$$

$$\sqrt{\varepsilon} p_0^\varepsilon = \sqrt{\varepsilon} p^\varepsilon(\cdot, 0) \longrightarrow 0 \text{ strongly in } L^2(\Omega).$$

"initial layer" phenomenon for the pressure initial datum

Artificial compressibility approximation

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \dfrac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \end{cases}$$

Boundary condition:

Artificial compressibility approximation

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \frac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \end{cases}$$

Boundary condition:

$$u^\varepsilon|_{\partial\Omega} = 0$$

Artificial compressibility approximation

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \frac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \end{cases}$$

Boundary condition:

$$u^\varepsilon|_{\partial\Omega} = 0$$

$$\varepsilon \int_0^t \int_\Omega p^\varepsilon(x,s)\phi(x,s)dxds + \int_0^t \int_\Omega u^\varepsilon(x,s)\nabla\phi(x,s)dxds$$

$$- \int_0^t \int_{\partial\Omega} (u^\varepsilon \cdot n)(x,s)\phi(x,s)d\sigma dt + \varepsilon \int_\Omega p_0^\varepsilon(x)\phi(x,0)dx = 0.$$

Artificial compressibility approximation

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \frac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \end{cases}$$

Boundary condition:

$$u^\varepsilon|_{\partial\Omega} = 0$$

$$\varepsilon \int_0^t \int_\Omega p^\varepsilon(x,s)\phi(x,s)dxds + \int_0^t \int_\Omega u^\varepsilon(x,s)\nabla\phi(x,s)dxds$$

$$- \int_0^t \int_{\partial\Omega} (u^\varepsilon \cdot n)(x,s)\phi(x,s)d\sigma dt + \varepsilon \int_\Omega p_0^\varepsilon(x)\phi(x,0)dx = 0$$

$$p^\varepsilon(x,t) = p_0^\varepsilon(x) \qquad \text{a.e. in } \partial\Omega$$

Artificial compressibility approximation

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla) u^\varepsilon - \frac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \end{cases}$$

Initial conditions

$u_0^\varepsilon = u^\varepsilon(\cdot, 0) \longrightarrow u_0 = u(\cdot, 0)$ strongly in $L^2(\Omega)$

$\sqrt{\varepsilon} p_0^\varepsilon = \sqrt{\varepsilon} p^\varepsilon(\cdot, 0) \longrightarrow 0$ strongly in $L^2(\Omega)$.

"initial layer" phenomenon for the pressure initial datum

Boundary conditions

$$u^\varepsilon(x,t) = 0 \qquad x \in \partial\Omega,\ t \geq 0$$

$$p^\varepsilon(x,t) = p_0^\varepsilon(x) \qquad x \in \partial\Omega,\ t \geq 0$$

Notations

Nonhomogenous Sobolev Spaces:

$$W^{k,p}(\Omega) = (I - \Delta)^{-\frac{k}{2}} L^p(\Omega) \qquad H^k(\Omega) = W^{k,2}(\Omega)$$

Notations

Nonhomogenous Sobolev Spaces:

$$W^{k,p}(\Omega) = (I - \Delta)^{-\frac{k}{2}} L^p(\Omega) \qquad H^k(\Omega) = W^{k,2}(\Omega)$$

$$L_t^p L_x^q = L^p([0,T]; L^q(\Omega)) \quad L_t^p W_x^{k,q} = L^p([0,T]; W^{k,q}(\Omega))$$

Notations

Nonhomogenous Sobolev Spaces:

$$W^{k,p}(\Omega) = (I - \Delta)^{-\frac{k}{2}} L^p(\Omega) \qquad H^k(\Omega) = W^{k,2}(\Omega)$$

$$L^p_t L^q_x = L^p([0,T]; L^q(\Omega)) \quad L^p_t W^{k,q}_x = L^p([0,T]; W^{k,q}(\Omega))$$

Leray Projectors

$Q = \nabla \Delta_N^{-1} \operatorname{div}$ projection on gradient vector fields

$P = I - Q$ projection on divergence - free vector fields

Main Theorem

Let $(u^\varepsilon, p^\varepsilon)$ be a sequence of weak solution in Ω of the previous system, then

(i) $u^\varepsilon \rightharpoonup u$ weakly in $L^2_t \dot{H}^1_x$

Main Theorem

Let $(u^\varepsilon, p^\varepsilon)$ be a sequence of weak solution in Ω of the previous system, then

(i) $u^\varepsilon \rightharpoonup u$ weakly in $L^2_t \dot{H}^1_x$

(ii) $Qu^\varepsilon \longrightarrow 0$ strongly in $L^2_t L^p_x$, for any $p \in [4, 6)$

Main Theorem

Let $(u^\varepsilon, p^\varepsilon)$ be a sequence of weak solution in Ω of the previous system, then

(i) $u^\varepsilon \rightharpoonup u$ weakly in $L^2_t \dot{H}^1_x$

(ii) $Qu^\varepsilon \longrightarrow 0$ strongly in $L^2_t L^p_x$, for any $p \in [4, 6)$

(iii) $Pu^\varepsilon \longrightarrow Pu = u$ strongly in $L^2_t L^2_{loc_x}$

Main Theorem

Let $(u^\varepsilon, p^\varepsilon)$ be a sequence of weak solution in Ω of the previous system, then

(i) $u^\varepsilon \rightharpoonup u$ weakly in $L_t^2 \dot{H}_x^1$

(ii) $Qu^\varepsilon \longrightarrow 0$ strongly in $L_t^2 L_x^p$, for any $p \in [4, 6)$

(iii) $Pu^\varepsilon \longrightarrow Pu = u$ strongly in $L_t^2 L_{loc_x}^2$

(iv) The sequence $\{p^\varepsilon\}$ will converge in the sense of distribution to

$$p = \Delta^{-1} \operatorname{div}((u \cdot \nabla) u) = \Delta^{-1} tr((Du)^2).$$

(v) $u = Pu$ is a Leray weak solution to the incompressible Navier Stokes equation

$$P(\partial_t u - \Delta u + (u \cdot \nabla)u) = 0 \quad \text{in } \mathcal{D}'([0,T] \times \Omega),$$

$$u(x,0) = u_0(x) \qquad u|_{\partial\Omega} = 0$$

(v) $u = Pu$ is a Leray weak solution to the incompressible Navier Stokes equation

$$P(\partial_t u - \Delta u + (u \cdot \nabla)u) = 0 \quad \text{in } \mathcal{D}'([0,T] \times \Omega),$$

$$u(x,0) = u_0(x) \qquad u|_{\partial\Omega} = 0$$

(vi) The following energy inequality holds

$$\frac{1}{2}\int_\Omega |u(x,t)|^2 dx + \int_0^T \int_\Omega |\nabla u(x,t)|^2 dx dt \le \frac{1}{2}\int_\Omega |u(x,0)|^2 dx.$$

(v) $u = Pu$ is a Leray weak solution to the incompressible Navier Stokes equation

$$P(\partial_t u - \Delta u + (u \cdot \nabla)u) = 0 \quad \text{in } \mathcal{D}'([0,T] \times \Omega),$$

$$u(x,0) = u_0(x) \qquad u|_{\partial\Omega} = 0$$

(vi) The following energy inequality holds

$$\frac{1}{2}\int_\Omega |u(x,t)|^2 dx + \int_0^T \int_\Omega |\nabla u(x,t)|^2 dx dt \leq \frac{1}{2}\int_\Omega |u(x,0)|^2 dx.$$

(!!) For u^ε the trace operator commutes with the limit, this is not true for p^ε.

References

- Artificial Compressibility in exterior domain

References

- Artificial Compressibility in exterior domain
  - D. Donatelli and P.Marcati, preprint

References

- Artificial Compressibility in exterior domain
  - D. Donatelli and P.Marcati, preprint
- Artificial Compressibility in $\mathbb{R}^3$

References

- Artificial Compressibility in exterior domain
  - D. Donatelli and P.Marcati, preprint
- Artificial Compressibility in $\mathbb{R}^3$
  - D. Donatelli P.Marcati, A dispersive approach to the artificial compressibility approximations of the Navier Stokes equation, *Journal of Hyperbolic Differential Equations*, 3, no. 3, (2006), 575-588.

References

- Artificial Compressibility in exterior domain
  - D. Donatelli and P.Marcati, preprint
- Artificial Compressibility in $\mathbb{R}^3$
  - D. Donatelli P.Marcati, A dispersive approach to the artificial compressibility approximations of the Navier Stokes equation, *Journal of Hyperbolic Differential Equations*, 3, no. 3, (2006), 575-588.
  - D. Donatelli, On the artificial compressibility method for the Navier Stokes Fourier system. *Preprint 2008 (submitted)*.

References

- Artificial Compressibility in exterior domain
  - D. Donatelli and P.Marcati, preprint
- Artificial Compressibility in $\mathbb{R}^3$
  - D. Donatelli P.Marcati, A dispersive approach to the artificial compressibility approximations of the Navier Stokes equation, *Journal of Hyperbolic Differential Equations*, 3, no. 3, (2006), 575-588.
  - D. Donatelli, On the artificial compressibility method for the Navier Stokes Fourier system.*Preprint 2008 (submitted)*.
- Artificial Compressibility in bounded domain

References

- Artificial Compressibility in exterior domain
  - D. Donatelli and P.Marcati, preprint
- Artificial Compressibility in $\mathbb{R}^3$
  - D. Donatelli P.Marcati, A dispersive approach to the artificial compressibility approximations of the Navier Stokes equation, *Journal of Hyperbolic Differential Equations*, 3, no. 3, (2006), 575-588.
  - D. Donatelli, On the artificial compressibility method for the Navier Stokes Fourier system.*Preprint 2008 (submitted)*.
- Artificial Compressibility in bounded domain
  - Numerical methods:
    Vladimirova, Kuznecov, Yanenko ('66), Chorin ('68, '69), Oskolkov ('71), Kuznecov and Smagulov ('75), Smagulov ('79), Yanenko, Kuznecov and Smagulov ('84)

References

- Artificial Compressibility in exterior domain
  - D. Donatelli and P.Marcati, preprint
- Artificial Compressibility in $\mathbb{R}^3$
  - D. Donatelli P.Marcati, A dispersive approach to the artificial compressibility approximations of the Navier Stokes equation, *Journal of Hyperbolic Differential Equations*, 3, no. 3, (2006), 575-588.
  - D. Donatelli, On the artificial compressibility method for the Navier Stokes Fourier system.*Preprint 2008 (submitted)*.
- Artificial Compressibility in bounded domain
  - Numerical methods:
    Vladimirova, Kuznecov, Yanenko ('66), Chorin ('68, '69), Oskolkov ('71), Kuznecov and Smagulov ('75), Smagulov ('79), Yanenko, Kuznecov and Smagulov ('84)
  - Rigorous convergence results:
    Ghidaglia and Temam ('88), Temam ('69, '01)

Energy estimates

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \dfrac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \end{cases}$$

$$E(t) = \int_\Omega \left(\frac{1}{2} |u^\varepsilon(x,t)|^2 + \frac{\varepsilon}{2} |p^\varepsilon(x,t)|^2 \right) dx$$

Energy estimates

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \dfrac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \end{cases}$$

$$E(t) = \int_\Omega \left(\frac{1}{2} |u^\varepsilon(x,t)|^2 + \frac{\varepsilon}{2} |p^\varepsilon(x,t)|^2 \right) dx$$

$$E(t) + \int_0^t \int_\Omega |\nabla u^\varepsilon(x,s)|^2 dx ds = E(0)$$

Energy estimates

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \frac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \end{cases}$$

$$E(t) = \int_\Omega \left(\frac{1}{2} |u^\varepsilon(x,t)|^2 + \frac{\varepsilon}{2} |p^\varepsilon(x,t)|^2 \right) dx$$

$$E(t) + \int_0^t \int_\Omega |\nabla u^\varepsilon(x,s)|^2 dx ds = E(0)$$

$$\Downarrow$$

Energy estimates

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \mu \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \frac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \end{cases}$$

$$E(t) = \int_\Omega \left(\frac{1}{2} |u^\varepsilon(x,t)|^2 + \frac{\varepsilon}{2} |p^\varepsilon(x,t)|^2 \right) dx$$

$$E(t) + \int_0^t \int_\Omega |\nabla u^\varepsilon(x,s)|^2 dx ds = E(0)$$

$$\Downarrow$$

$\sqrt{\varepsilon} p^\varepsilon$ bd. in $L^\infty_t L^2_x$,

$\varepsilon p^\varepsilon_t$ relatively compact in $H^{-1}_{t,x}$

∇u^ε bd. in $L^2_{t,x}$,

u^ε bd. in $L^\infty_t L^2_x \cap L^2_t L^6_x$,

$(u^\varepsilon \cdot \nabla) u^\varepsilon$ bd. in $L^2_t L^1_x \cap L^1_t L^{3/2}_x$,

$(\operatorname{div} u^\varepsilon) u^\varepsilon$ bd. in $L^2_t L^1_x \cap L^1_t L^{3/2}_x$

Estimates on Qu^ε - Part 1

$$Qu^\varepsilon = \nabla \Delta_N^{-1} \operatorname{div} u^\varepsilon$$

Estimates on Qu^ε - Part 1

$$Qu^\varepsilon = \nabla \Delta_N^{-1} \operatorname{div} u^\varepsilon \qquad \varepsilon \partial_t p^\varepsilon = - \operatorname{div} u^\varepsilon$$

Estimates on Qu^ε - Part 1

$$Qu^\varepsilon = \nabla \Delta_N^{-1} \operatorname{div} u^\varepsilon \qquad \varepsilon \partial_t p^\varepsilon = -\operatorname{div} u^\varepsilon$$

$$Qu^\varepsilon = \varepsilon \nabla \Delta_N^{-1} \partial_t p^\varepsilon$$

Pressure wave equation

Differentiate in t the "pressure equation"

$$\varepsilon \partial_{tt} p^{\varepsilon} + \operatorname{div} \partial_t u^{\varepsilon} = 0$$

Pressure wave equation

Differentiate in t the "pressure equation"

$$\varepsilon\partial_{tt}p^{\varepsilon} + \operatorname{div}\partial_t u^{\varepsilon} = 0$$

taking the divergence of the first equation

$$\varepsilon\partial_{tt}p^{\varepsilon} - \Delta p^{\varepsilon} = -\Delta \operatorname{div} u^{\varepsilon} + \operatorname{div}\left((u^{\varepsilon}\cdot\nabla)\, u^{\varepsilon} + \frac{1}{2}(\operatorname{div} u^{\varepsilon})u^{\varepsilon} \right)$$

Pressure wave equation

Differentiate in t the "pressure equation"

$$\varepsilon \partial_{tt} p^{\varepsilon} + \operatorname{div} \partial_t u^{\varepsilon} = 0$$

taking the divergence of the first equation

$$\varepsilon \partial_{tt} p^{\varepsilon} - \Delta p^{\varepsilon} = -\Delta \operatorname{div} u^{\varepsilon} + \operatorname{div} \left((u^{\varepsilon} \cdot \nabla) u^{\varepsilon} + \frac{1}{2} (\operatorname{div} u^{\varepsilon}) u^{\varepsilon} \right)$$

changing the time scale: $t = \sqrt{\varepsilon} \tau$

$$\tilde{u}(x, \tau) = u^{\varepsilon}(x, \sqrt{\varepsilon} \tau), \quad \bar{p}(x, \tau) = p^{\varepsilon}(x, \sqrt{\varepsilon} \tau)$$

$$\partial_{\tau\tau} \bar{p} - \Delta \bar{p} = - \Delta \operatorname{div} \tilde{u} + \operatorname{div} \left((\tilde{u} \cdot \nabla) \tilde{u} + \frac{1}{2} (\operatorname{div} \tilde{u}) \tilde{u} \right)$$

Pressure wave equation

Differentiate in t the "pressure equation"

$$\varepsilon\partial_{tt}p^{\varepsilon} + \operatorname{div}\partial_t u^{\varepsilon} = 0$$

taking the divergence of the first equation

$$\varepsilon\partial_{tt}p^{\varepsilon} - \Delta p^{\varepsilon} = -\Delta\operatorname{div} u^{\varepsilon} + \operatorname{div}\left(\left(u^{\varepsilon}\cdot\nabla\right)u^{\varepsilon} + \frac{1}{2}(\operatorname{div} u^{\varepsilon})u^{\varepsilon}\right)$$

changing the time scale: $t = \sqrt{\varepsilon}\tau$

$$\tilde{u}(x,\tau) = u^{\varepsilon}(x,\sqrt{\varepsilon}\tau), \quad \bar{p}(x,\tau) = p^{\varepsilon}(x,\sqrt{\varepsilon}\tau)$$

$$\partial_{\tau\tau}\bar{p} - \Delta\bar{p} = -\Delta\underbrace{\operatorname{div}\tilde{u}}_{L^2_{t,x}} - \operatorname{div}\underbrace{\left(\left(\tilde{u}\cdot\nabla\right)\tilde{u} + \frac{1}{2}(\operatorname{div}\tilde{u})\tilde{u}\right)}_{L^1_t L^{3/2}_x}$$

Strichartz Estimates

$$\begin{cases} w_{tt} - \Delta w = F \\ w(0,\cdot) = f \qquad\qquad\qquad (x,t) \in \mathbb{R}^d \times [0,T] \\ \partial_t w(0,\cdot) = g \end{cases}$$

Strichartz Estimates

$$\begin{cases} w_{tt} - \Delta w = F \\ w(0,\cdot) = f \qquad\qquad (x,t) \in \mathbb{R}^d \times [0,T] \\ \partial_t w(0,\cdot) = g \end{cases}$$

$$\sup_{t\in[0,T]} \left(\|w\|_{\dot{H}^1_x} + \|w_t\|_{L^2_x}\right) \leq \|f\|_{\dot{H}^1_x} + \|g\|_{L^2_x} + \int_0^T \|F\|_{L^2_x} ds.$$

Strichartz Estimates

$$\begin{cases} w_{tt} - \Delta w = F \\ w(0,\cdot) = f \qquad\qquad (x,t) \in \mathbb{R}^d \times [0,T] \\ \partial_t w(0,\cdot) = g \end{cases}$$

$$\sup_{t\in[0,T]} \left(\|w\|_{\dot{H}^1_x} + \|w_t\|_{L^2_x} \right) \leq \|f\|_{\dot{H}^1_x} + \|g\|_{L^2_x} + \int_0^T \|F\|_{L^2_x} ds.$$

we would like to have

$$\|w\|_{L^q_t L^r_x} \leq \|f\|_{\dot{H}^\gamma_x} + \|g\|_{\dot{H}^{\gamma-1}_x} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \quad \gamma \text{ small}$$

Strichartz Estimates

$$\begin{cases} w_{tt} - \Delta w = F \\ w(0,\cdot) = f \qquad\qquad (x,t) \in \mathbb{R}^d \times [0,T] \\ \partial_t w(0,\cdot) = g \end{cases}$$

$$\sup_{t\in[0,T]} \left(\|w\|_{\dot{H}^1_x} + \|w_t\|_{L^2_x}\right) \leq \|f\|_{\dot{H}^1_x} + \|g\|_{L^2_x} + \int_0^T \|F\|_{L^2_x} ds.$$

we would like to have

$$\|w\|_{L^q_t L^r_x} \leq \|f\|_{\dot{H}^\gamma_x} + \|g\|_{\dot{H}^{\gamma-1}_x} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \quad \gamma \text{ small}$$

Strichartz (1977) proved that

$$\|w\|_{L^4_{t,x}} \leq \|f\|_{\dot{H}^{1/2}_x} + \|g\|_{\dot{H}^{-1/2}_x} + \|F\|_{L^{4/3}_{t,x}}$$

Idea of the proof

Strichartz realized that these estimates are related to the restriction theorem for the Fourier transform.

Idea of the proof

Strichartz realized that these estimates are related to the restriction theorem for the Fourier transform.

Fourier transform: $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} dx$

Restriction mapping: $Rf = \hat{f}|_S \quad S \subset \mathbb{R}^d$ hypersurface

Idea of the proof

Strichartz realized that these estimates are related to the restriction theorem for the Fourier transform.

Fourier transform: $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} dx$

Restriction mapping: $Rf = \hat{f}|_S \quad S \subset \mathbb{R}^d$ hypersurface

S has the $(p, 2)$ restriction property if

$$\|Rf\|_{L^2(S)} \leq \|f\|_{L^p(\mathbb{R}^d)}.$$

Idea of the proof

Strichartz realized that these estimates are related to the restriction theorem for the Fourier transform.

Fourier transform: $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} dx$

Restriction mapping: $Rf = \hat{f}|_S$ $\quad S \subset \mathbb{R}^d$ hypersurface

S has the $(p, 2)$ restriction property if

$$\|Rf\|_{L^2(S)} \leq \|f\|_{L^p(\mathbb{R}^d)}.$$

If $S =$ sphere $\subset \mathbb{R}^3$, then $1 \leq p \leq \frac{4}{3}$

$$\|w\|_{L_t^q L_x^r} \leq \|f\|_{\dot{H}_x^\gamma}$$

then if β is a cutoff function (localizing in frequencies)

$$\|w\|_{L_t^q L_x^r} \leq \|f\|_{\dot{H}_x^\gamma}$$

then if β is a cutoff function (localizing in frequencies)

$$w(t,x) = \int_{\mathbb{R}^d} e^{ix\xi} e^{it|\xi|} \beta(\xi) \hat{f}(\xi) d\xi := Tf(t,x)$$

$$\|w\|_{L^q_t L^r_x} \leq \|f\|_{\dot{H}^\gamma_x}$$

then if β is a cutoff function (localizing in frequencies)

$$w(t,x) = \int_{\mathbb{R}^d} e^{ix\xi} e^{it|\xi|} \beta(\xi) \hat{f}(\xi) d\xi := Tf(t,x)$$

$$\|w\|_{L^q_t L^r_x} \leq \|f\|_{L^2} \Longleftrightarrow \|Tf\|_{L^q_t L^r_x} \leq \|f\|_{L^2} \quad (*)$$

$$T : L^2_\xi \to L^q_t L^r_x \Longleftrightarrow T^* : L^{q'}_t L^{r'}_x \to L^2_\xi \Longleftrightarrow TT^* : L^{q'}_t L^{r'}_x \to L^q_t L^r_x$$

T^* is the adjoint operator of T, (q', r') are the dual exponent of (q, r)

$$\|w\|_{L^q_t L^r_x} \leq \|f\|_{\dot{H}^\gamma_x}$$

then if β is a cutoff function (localizing in frequencies)

$$w(t,x) = \int_{\mathbb{R}^d} e^{ix\xi} e^{it|\xi|} \beta(\xi) \hat{f}(\xi) d\xi := Tf(t,x)$$

$$\|w\|_{L^q_t L^r_x} \leq \|f\|_{L^2} \iff \|Tf\|_{L^q_t L^r_x} \leq \|f\|_{L^2} \quad (*)$$

$$T : L^2_\xi \to L^q_t L^r_x \iff T^* : L^{q'}_t L^{r'}_x \to L^2_\xi \iff TT^* : L^{q'}_t L^{r'}_x \to L^q_t L^r_x$$

T^* is the adjoint operator of T, (q', r') are the dual exponent of (q, r)

$$T^* f(x) = \int_{\mathbb{R}^d} e^{ix\xi} \overline{\beta(\xi)} \tilde{f}(|\xi|, \xi) d\xi \quad \widehat{T^* f}(\xi) \simeq \overline{\beta(\xi)} \tilde{f}(|\xi|, \xi) = Rf(\xi)$$

$$\|w\|_{L^q_t L^r_x} \leq \|f\|_{\dot{H}^\gamma_x}$$

then if β is a cutoff function (localizing in frequencies)

$$w(t,x) = \int_{\mathbb{R}^d} e^{ix\xi} e^{it|\xi|} \beta(\xi) \hat{f}(\xi) d\xi := Tf(t,x)$$

$$\|w\|_{L^q_t L^r_x} \leq \|f\|_{L^2} \Longleftrightarrow \|Tf\|_{L^q_t L^r_x} \leq \|f\|_{L^2} \quad (*)$$

$$T : L^2_\xi \to L^q_t L^r_x \Longleftrightarrow T^* : L^{q'}_t L^{r'}_x \to L^2_\xi \Longleftrightarrow TT^* : L^{q'}_t L^{r'}_x \to L^q_t L^r_x$$

T^* is the adjoint operator of T, (q', r') are the dual exponent of (q, r)

$$T^* f(x) = \int_{\mathbb{R}^d} e^{ix\xi} \overline{\beta(\xi)} \tilde{f}(|\xi|, \xi) d\xi \quad \widehat{T^* f}(\xi) \simeq \overline{\beta(\xi)} \tilde{f}(|\xi|, \xi) = Rf(\xi)$$

$\Lambda = \{(\tau, \xi) \mid \tau = |\xi| > 0\}$=light cone $\quad \|T^* f\|_{L^2} = \|Rf\|_{L^2(\Lambda)}$

$$\|w\|_{L_t^q L_x^r} \le \|f\|_{\dot H_x^\gamma}$$

then if β is a cutoff function (localizing in frequencies)

$$w(t,x) = \int_{\mathbb{R}^d} e^{ix\xi} e^{it|\xi|} \beta(\xi) \hat f(\xi) d\xi := Tf(t,x)$$

$$\|w\|_{L_t^q L_x^r} \le \|f\|_{L^2} \iff \|Tf\|_{L_t^q L_x^r} \le \|f\|_{L^2} \ (*)$$

$$T : L^2_\xi \to L_t^q L_x^r \iff T^* : L_t^{q'} L_x^{r'} \to L^2_\xi \iff TT^* : L_t^{q'} L_x^{r'} \to L_t^q L_x^r$$

T^* is the adjoint operator of T, (q', r') are the dual exponent of (q, r)

$$T^* f(x) = \int_{\mathbb{R}^d} e^{ix\xi} \overline{\beta(\xi)} \tilde f(|\xi|, \xi) d\xi \quad \widehat{T^* f}(\xi) \simeq \overline{\beta(\xi)} \tilde f(|\xi|, \xi) = Rf(\xi)$$

$\Lambda = \{(\tau, \xi) \mid \tau = |\xi| > 0\}$=light cone $\quad \|T^* f\|_{L^2} = \|Rf\|_{L^2(\Lambda)}$

$(*)$ is equivalent to $R : L_t^{q'} L_x^{r'} \to L^2(\Lambda)$ is bounded for suitable (q,r)

Strichartz Estimates (Ginibre-Velo ('95), Keel-Tao('98))

$$w_{tt} - \Delta w = F \quad w(0,\cdot) = f, \quad \partial_t w(0,\cdot) = g, \quad (x,t) \in \mathbb{R}^d \times [0,T]$$

Strichartz Estimates (Ginibre-Velo ('95), Keel-Tao('98))

$$w_{tt} - \Delta w = F \quad w(0,\cdot) = f, \quad \partial_t w(0,\cdot) = g, \quad (x,t) \in \mathbb{R}^d \times [0,T]$$

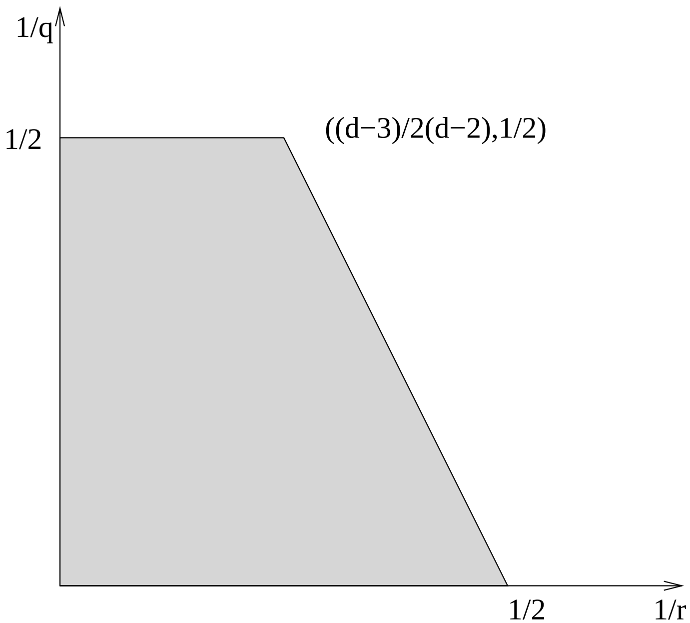
$$\|w\|_{L_t^q L_x^r} + \|\partial_t w\|_{L_t^q W_x^{-1,r}} \lesssim \|f\|_{\dot{H}_x^{\gamma}} + \|g\|_{\dot{H}_x^{\gamma-1}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

Strichartz Estimates (Ginibre-Velo ('95), Keel-Tao('98))

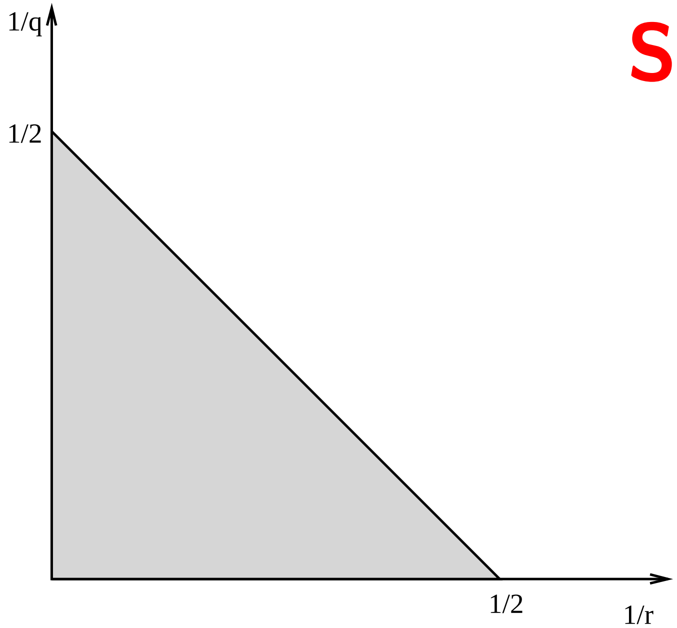
$$w_{tt} - \Delta w = F \quad w(0,\cdot) = f, \quad \partial_t w(0,\cdot) = g, \quad (x,t) \in \mathbb{R}^d \times [0,T]$$

$$\|w\|_{L_t^q L_x^r} + \|\partial_t w\|_{L_t^q W_x^{-1,r}} \lesssim \|f\|_{\dot{H}_x^\gamma} + \|g\|_{\dot{H}_x^{\gamma-1}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

(q,r), $(\tilde{q},\tilde{r})$ are *wave admissible pairs* if



$$q, r, \tilde{q}, \tilde{r} \geq 2$$
$$\frac{2}{q} \leq (d-1)\left(\frac{1}{2} - \frac{1}{r}\right)$$
$$\frac{2}{\tilde{q}} \leq (d-1)\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right)$$
$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \gamma = \frac{1}{\tilde{q}'} + \frac{d}{\tilde{r}'} - 2$$

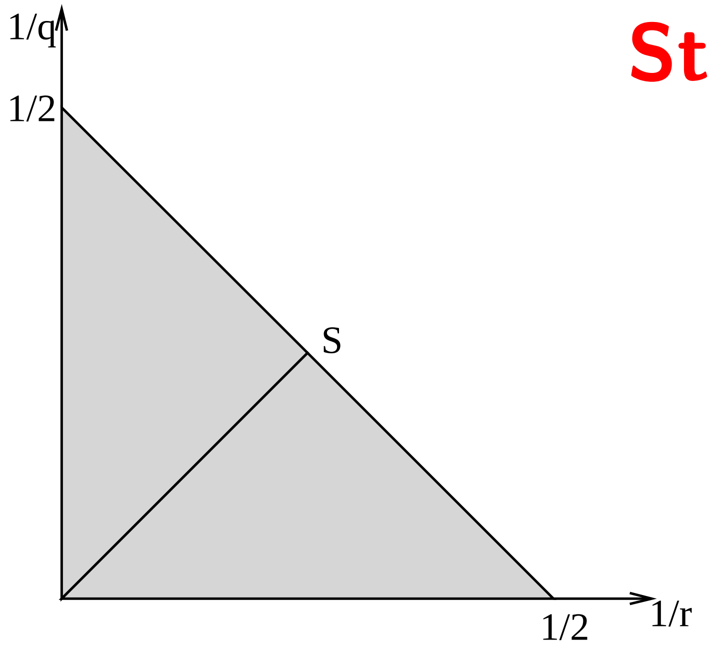


Strichartz d=3

$$q, r, \tilde{q}, \tilde{r} \geq 2$$
$$\frac{1}{q} \leq \frac{1}{2} - \frac{1}{r}$$
$$\frac{1}{\tilde{q}} \leq \frac{1}{2} - \frac{1}{\tilde{r}}$$
$$\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - \gamma = \frac{1}{\tilde{q}'} + \frac{3}{\tilde{r}'} - 2$$

$$\|w\|_{L_t^q L_x^r} + \|\partial_t w\|_{L_t^q W_x^{-1,r}} \lesssim \|f\|_{\dot{H}_x^\gamma} + \|g\|_{\dot{H}_x^{\gamma-1}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

Strichartz d=3



$$q, r, \tilde{q}, \tilde{r} \geq 2$$

$$\frac{1}{q} \leq \frac{1}{2} - \frac{1}{r}$$

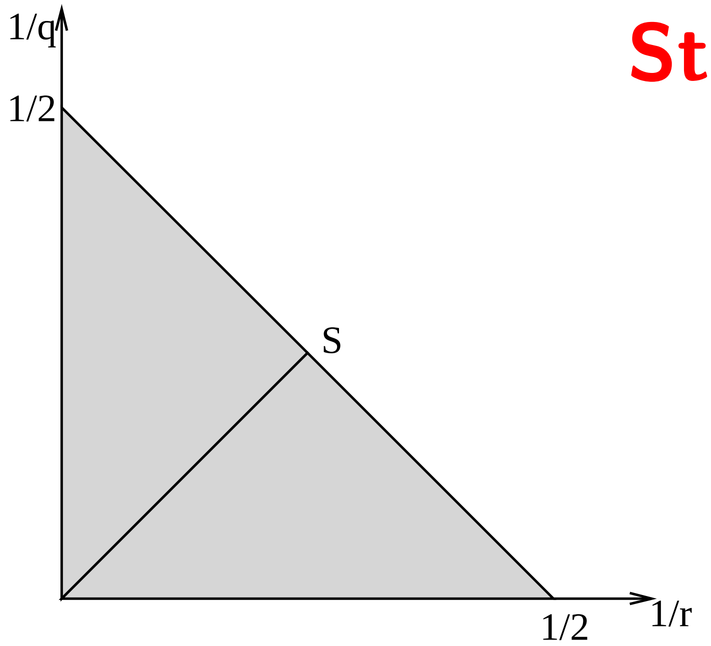
$$\frac{1}{\tilde{q}} \leq \frac{1}{2} - \frac{1}{\tilde{r}}$$

$$\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - \gamma = \frac{1}{\tilde{q}'} + \frac{3}{\tilde{r}'} - 2$$

$$\|w\|_{L_t^q L_x^r} + \|\partial_t w\|_{L_t^q W_x^{-1,r}} \lesssim \|f\|_{\dot{H}_x^\gamma} + \|g\|_{\dot{H}_x^{\gamma-1}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

$$\|w\|_{L_{t,x}^4} + \|\partial_t w\|_{L_t^4 W_x^{-1,4}} \lesssim \|f\|_{\dot{H}_x^{1/2}} + \|g\|_{\dot{H}_x^{-1/2}} + \|F\|_{L_{t,x}^{4/3}}$$

Strichartz d=3



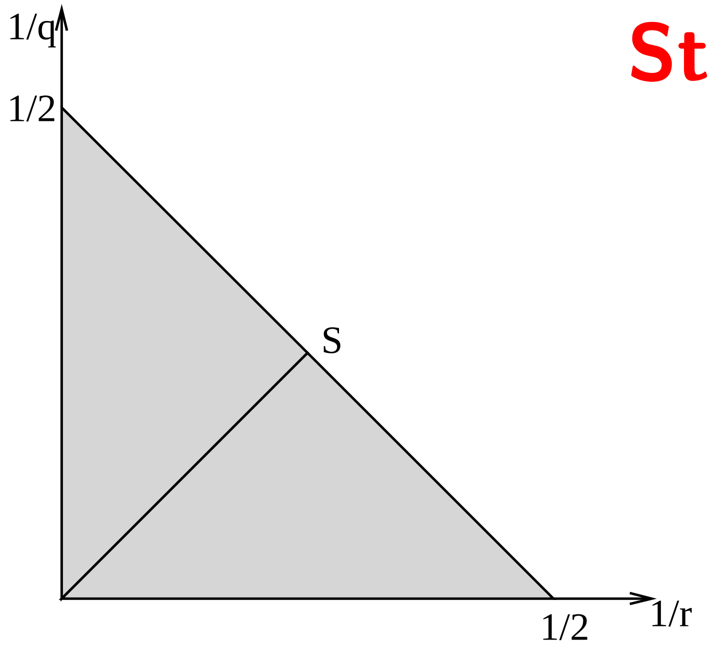
$$q, r, \tilde{q}, \tilde{r} \geq 2$$
$$\frac{1}{q} \leq \frac{1}{2} - \frac{1}{r}$$
$$\frac{1}{\tilde{q}} \leq \frac{1}{2} - \frac{1}{\tilde{r}}$$
$$\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - \gamma = \frac{1}{\tilde{q}'} + \frac{3}{\tilde{r}'} - 2$$

$$\|w\|_{L_t^q L_x^r} + \|\partial_t w\|_{L_t^q W_x^{-1,r}} \lesssim \|f\|_{\dot{H}_x^\gamma} + \|g\|_{\dot{H}_x^{\gamma-1}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}},$$

$$\|w\|_{L^4_{t,x}} + \|\partial_t w\|_{L_t^4 W_x^{-1,4}} \lesssim \|f\|_{\dot{H}_x^{1/2}} + \|g\|_{\dot{H}_x^{-1/2}} + \|F\|_{L^{4/3}_{t,x}}$$

if $d = 3$, $(\tilde{q}', \tilde{r}') = (1, 3/2)$, then $\gamma = 1/2$ and $(q, r) = (4, 4)$

Strichartz d=3



$$q, r, \tilde{q}, \tilde{r} \geq 2$$
$$\frac{1}{q} \leq \frac{1}{2} - \frac{1}{r}$$
$$\frac{1}{\tilde{q}} \leq \frac{1}{2} - \frac{1}{\tilde{r}}$$
$$\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - \gamma = \frac{1}{\tilde{q}'} + \frac{3}{\tilde{r}'} - 2$$

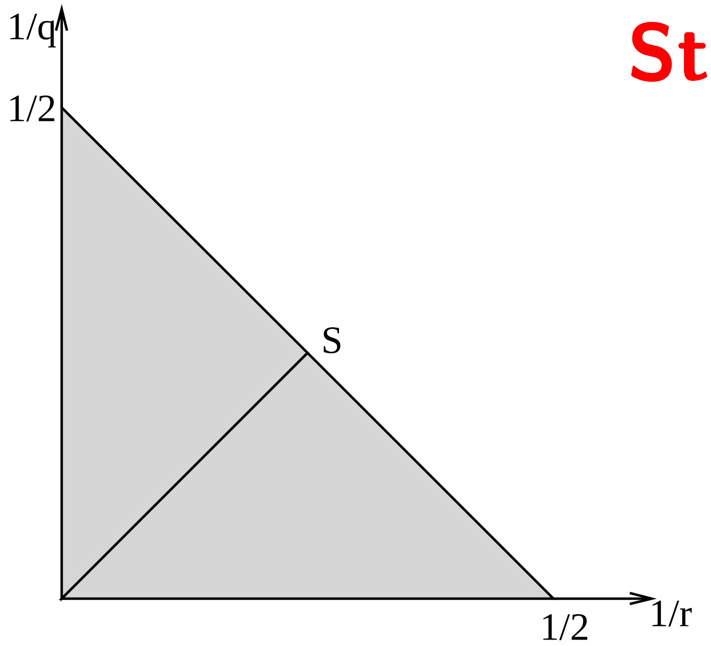
$$\|w\|_{L_t^q L_x^r} + \|\partial_t w\|_{L_t^q W_x^{-1,r}} \lesssim \|f\|_{\dot{H}_x^\gamma} + \|g\|_{\dot{H}_x^{\gamma-1}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}},$$

$$\|w\|_{L_{t,x}^4} + \|\partial_t w\|_{L_t^4 W_x^{-1,4}} \lesssim \|f\|_{\dot{H}_x^{1/2}} + \|g\|_{\dot{H}_x^{-1/2}} + \|F\|_{L_{t,x}^{4/3}}$$

if $d = 3$, $(\tilde{q}', \tilde{r}') = (1, 3/2)$, then $\gamma = 1/2$ and $(q, r) = (4, 4)$

$$\|w\|_{L_{t,x}^4} + \|\partial_t w\|_{L_t^4 W_x^{-1,4}} \lesssim \|f\|_{\dot{H}_x^{1/2}} + \|g\|_{\dot{H}_x^{-1/2}} + \|F\|_{L_t^1 L_x^{3/2}}$$

Strichartz d=3



$$q, r, \tilde{q}, \tilde{r} \geq 2$$
$$\frac{1}{q} \leq \frac{1}{2} - \frac{1}{r}$$
$$\frac{1}{\tilde{q}} \leq \frac{1}{2} - \frac{1}{\tilde{r}}$$
$$\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - \gamma = \frac{1}{\tilde{q}'} + \frac{3}{\tilde{r}'} - 2$$

$$\|w\|_{L^q_t L^r_x} + \|\partial_t w\|_{L^q_t W^{-1,r}_x} \lesssim \|f\|_{\dot{H}^\gamma_x} + \|g\|_{\dot{H}^{\gamma-1}_x} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x},$$

$$\|w\|_{L^4_{t,x}} + \|\partial_t w\|_{L^4_t W^{-1,4}_x} \lesssim \|f\|_{\dot{H}^{1/2}_x} + \|g\|_{\dot{H}^{-1/2}_x} + \|F\|_{L^{4/3}_{t,x}}$$

if $d = 3$, $(\tilde{q}', \tilde{r}') = (1, 3/2)$, then $\gamma = 1/2$ and $(q, r) = (4, 4)$

$$\|w\|_{L^4_{t,x}} + \|\partial_t w\|_{L^4_t W^{-1,4}_x} \lesssim \|f\|_{\dot{H}^{1/2}_x} + \|g\|_{\dot{H}^{-1/2}_x} + \|F\|_{L^1_t L^{3/2}_x}$$

$$\|w\|_{L^4_{t,x}} + \|\partial_t w\|_{L^4_t W^{-1,4}_x} \lesssim \|f\|_{\dot{H}^{1/2}_x} + \|g\|_{\dot{H}^{1/2}_x} + \|F\|_{L^1_t L^2_x}.$$

Strichartz estimates on exterior domain Ω

(Smith, Sogge, Metcalf, Burq)

$$\begin{cases} \left(\partial_t^2 - \Delta\right) w(t,x) = F(t,x), & (t,x) \in \mathbb{R}_+ \times \Omega \\ w(0,\cdot) = f(x) \in \dot{H}_D^{\gamma} & \\ \partial_t w(0,x) = g(x) \in \dot{H}_D^{\gamma-1} & \\ w(t,x) = 0, & x \in \partial\Omega, \end{cases}$$

$$\|w\|_{L_t^q L_x^r} + \|\partial_t w\|_{L_t^q W_x^{-1,r}} \lesssim \|f\|_{\dot{H}_D^{\gamma}} + \|g\|_{\dot{H}_D^{\gamma-1}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

Strichartz estimates on exterior domain Ω

(Smith, Sogge, Metcalf, Burq)

$$\begin{cases} \left(\partial_t^2 - \Delta\right) w(t,x) = F(t,x), & (t,x) \in \mathbb{R}_+ \times \Omega \\ w(0,\cdot) = f(x) \in \dot{H}_D^{\gamma} & \\ \partial_t w(0,x) = g(x) \in \dot{H}_D^{\gamma-1} & \\ w(t,x) = 0, & x \in \partial\Omega, \end{cases}$$

$$\|w\|_{L_t^q L_x^r} + \|\partial_t w\|_{L_t^q W_x^{-1,r}} \lesssim \|f\|_{\dot{H}_D^{\gamma}} + \|g\|_{\dot{H}_D^{\gamma-1}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

$$\beta \in C_0^{\infty}(\mathbb{R}^d), \quad \beta(x) = 1 \text{ on } \{|x| \le R\}$$

$$\|f\|_{\dot{H}_D^{\gamma}} = \|\beta f\|_{H^{\gamma}(\Omega)} + \|(1-\beta) f\|_{\dot{H}^{\gamma}(\mathbb{R}^d)} \quad \Delta^j f|_{\partial\Omega} = 0, \ 2j < \gamma$$

Strichartz estimates on exterior domain Ω

(Smith, Sogge, Metcalf, Burq)

$$\begin{cases} \left(\partial_t^2 - \Delta\right) w(t,x) = F(t,x), & (t,x) \in \mathbb{R}_+ \times \Omega \\ w(0,\cdot) = f(x) \in \dot{H}_D^{\gamma} \\ \partial_t w(0,x) = g(x) \in \dot{H}_D^{\gamma-1} \\ w(t,x) = 0, & x \in \partial\Omega, \end{cases}$$

$$\|w\|_{L_t^q L_x^r} + \|\partial_t w\|_{L_t^q W_x^{-1,r}} \lesssim \|f\|_{\dot{H}_D^{\gamma}} + \|g\|_{\dot{H}_D^{\gamma-1}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

$$\beta \in C_0^{\infty}(\mathbb{R}^d), \quad \beta(x) = 1 \text{ on } \{|x| \leq R\}$$

$$\|f\|_{\dot{H}_D^{\gamma}} = \|\beta f\|_{H^{\gamma}(\Omega)} + \|(1-\beta) f\|_{\dot{H}^{\gamma}(\mathbb{R}^d)} \quad \Delta^j f|_{\partial\Omega} = 0, \ 2j < \gamma$$

Ω is nontrapping, there is L_R, such that non geodesic of lenght L_R is completely contained in $\{|x| \leq R\} \cap \Omega$

Sketch of the proof

- Smith and Sogge (1995) proved local Strichartz estimates

Sketch of the proof

- Smith and Sogge (1995) proved local Strichartz estimates
- Smith and Sogge (2000) proved global Strichartz estimates in odd space dimension

Sketch of the proof

- Smith and Sogge (1995) proved local Strichartz estimates
- Smith and Sogge (2000) proved global Strichartz estimates in odd space dimension
  - *exponential* decay of the local energy of solutions of the wave equation with compactly supported initial data

$$\|\beta u\|_{H^\gamma_D(\Omega)} + \|\beta \partial_t u\|_{H^{\gamma-1}_D(\Omega)} \leq C e^{-\alpha|t|} (\|f\|_{H^\gamma_D(\Omega)} + \|g\|_{H^\gamma_D(\Omega)})$$

Sketch of the proof

- Smith and Sogge (1995) proved local Strichartz estimates
- Smith and Sogge (2000) proved global Strichartz estimates in odd space dimension
  - *exponential* decay of the local energy of solutions of the wave equation with compactly supported initial data

$$\|\beta u\|_{H^\gamma_D(\Omega)}+\|\beta\partial_t u\|_{H^{\gamma-1}_D(\Omega)} \leq Ce^{-\alpha|t|}(\|f\|_{H^\gamma_D(\Omega)}+\|g\|_{H^\gamma_D(\Omega)})$$

- Burq (2004), Metcalfe (2003) proved global Strichartz estimates in even space dimension

Sketch of the proof

- Smith and Sogge (1995) proved local Strichartz estimates
- Smith and Sogge (2000) proved global Strichartz estimates in odd space dimension
  - *exponential* decay of the local energy of solutions of the wave equation with compactly supported initial data

$$\|\beta u\|_{H^\gamma_D(\Omega)}+\|\beta\partial_t u\|_{H^{\gamma-1}_D(\Omega)} \leq Ce^{-\alpha|t|}(\|f\|_{H^\gamma_D(\Omega)}+\|g\|_{H^\gamma_D(\Omega)})$$

- Burq (2004), Metcalfe (2003) proved global Strichartz estimates in even space dimension
  - Local energy decay

$$\|\beta u\|_{H^\gamma_D(\Omega)}+\|\beta\partial_t u\|_{H^{\gamma-1}_D(\Omega)} \leq C|t|^{-d/2}(\|f\|_{H^\gamma_D(\Omega)}+\|g\|_{H^\gamma_D(\Omega)})$$

Pressure wave equation

$$
\begin{cases}
\partial_{\tau\tau}\bar{p} - \Delta\bar{p} = -\Delta \operatorname{div}\tilde{u} + \operatorname{div}\left((\tilde{u}\cdot\nabla)\,\tilde{u} + \frac{1}{2}(\operatorname{div}\tilde{u})\tilde{u}\right), \\
\bar{p}(x,0) = p_0^{\varepsilon}(x), \\
\partial_\tau \bar{p}(x,0) = \varepsilon^{-1/2}\operatorname{div} u_0^{\varepsilon}(x), \\
\bar{p}(x,t)|_{\partial\Omega} = p_0^{\varepsilon}(x)|_{\partial\Omega}
\end{cases}
$$

for fixed ε smoothing of initial data

Pressure wave equation

$$\begin{cases} \partial_{\tau\tau}\bar{p} - \Delta\bar{p} = -\Delta\,\mathrm{div}\,\tilde{u} + \mathrm{div}\left((\tilde{u}\cdot\nabla)\,\tilde{u} + \frac{1}{2}(\mathrm{div}\,\tilde{u})\tilde{u}\right), \\ \bar{p}(x,0) = p_0^{\varepsilon}(x), \\ \partial_{\tau}\bar{p}(x,0) = \varepsilon^{-1/2}\,\mathrm{div}\,u_0^{\varepsilon}(x), \\ \bar{p}(x,t)|_{\partial\Omega} = p_0^{\varepsilon}(x)|_{\partial\Omega} \end{cases}$$

for fixed ε smoothing of initial data

we need homogenous boundary conditions:

Pressure wave equation

$$\begin{cases} \partial_{\tau\tau}\bar{p} - \Delta\bar{p} = -\Delta \operatorname{div} \tilde{u} + \operatorname{div}\left((\tilde{u}\cdot\nabla)\,\tilde{u} + \frac{1}{2}(\operatorname{div}\tilde{u})\tilde{u}\right), \\ \bar{p}(x,0) = p_0^{\varepsilon}(x), \\ \partial_\tau \bar{p}(x,0) = \varepsilon^{-1/2} \operatorname{div} u_0^{\varepsilon}(x), \\ \bar{p}(x,t)|_{\partial\Omega} = p_0^{\varepsilon}(x)|_{\partial\Omega} \end{cases}$$

for fixed ε smoothing of initial data
we need homogenous boundary conditions:

$$\tilde{p} = \bar{p} - p_0^{\varepsilon}$$

Pressure wave equation

$$\begin{cases} \partial_{\tau\tau}\bar{p} - \Delta\bar{p} = -\Delta\,\mathrm{div}\,\tilde{u} + \mathrm{div}\left((\tilde{u}\cdot\nabla)\,\tilde{u} + \frac{1}{2}(\mathrm{div}\,\tilde{u})\tilde{u}\right), \\ \bar{p}(x,0) = p_0^{\varepsilon}(x), \\ \partial_{\tau}\bar{p}(x,0) = \varepsilon^{-1/2}\,\mathrm{div}\,u_0^{\varepsilon}(x), \\ \bar{p}(x,t)|_{\partial\Omega} = p_0^{\varepsilon}(x)|_{\partial\Omega} \end{cases}$$

for fixed ε smoothing of initial data

we need homogenous boundary conditions:

$$\tilde{p} = \bar{p} - p_0^{\varepsilon}$$

$$\begin{cases} \partial_{\tau\tau}\tilde{p} - \Delta\tilde{p} = -\Delta\,\mathrm{div}\,\tilde{u} + \mathrm{div}\,\nabla p_0^{\varepsilon} + \mathrm{div}\left((\tilde{u}\cdot\nabla)\,\tilde{u} + \frac{1}{2}(\mathrm{div}\,\tilde{u})\tilde{u}\right) \\ \tilde{p}(x,0) = \partial_{\tau}\tilde{p}(x,0) = 0 \\ \tilde{p}|_{\partial\Omega} = 0. \end{cases}$$

Pressure wave equation

$$\begin{cases} \partial_{\tau\tau}\bar{p} - \Delta\bar{p} = -\Delta \operatorname{div}\tilde{u} + \operatorname{div}\Big((\tilde{u}\cdot\nabla)\,\tilde{u} + \frac{1}{2}(\operatorname{div}\tilde{u})\tilde{u}\Big)\,, \\ \bar{p}(x,0) = p_0^{\varepsilon}(x), \\ \partial_\tau\bar{p}(x,0) = \varepsilon^{-1/2}\operatorname{div} u_0^{\varepsilon}(x), \\ \bar{p}(x,t)|_{\partial\Omega} = p_0^{\varepsilon}(x)|_{\partial\Omega} \end{cases}$$

for fixed ε smoothing of initial data

we need homogenous boundary conditions:

$$\tilde{p} = \bar{p} - p_0^{\varepsilon}$$

$$\begin{cases} \partial_{\tau\tau}\tilde{p} - \Delta\tilde{p} = -\Delta\underbrace{\operatorname{div}\tilde{u}}_{L^2_{t,x}} + \operatorname{div}\nabla\underbrace{p_0^{\varepsilon}}_{L^2_x} + \underbrace{\operatorname{div}\left((\tilde{u}\cdot\nabla)\,\tilde{u} + \frac{1}{2}(\operatorname{div}\tilde{u})\tilde{u}\right)}_{L^1_t L^{3/2}_x} \\ \tilde{p}(x,0) = \partial_\tau\tilde{p}(x,0) = 0 \\ \tilde{p}|_{\partial\Omega} = 0. \end{cases}$$

$\tilde{p} = \tilde{p}_1 + \tilde{p}_2$ where $\tilde{p}_1$ and $\tilde{p}_2$ solve the following wave equations:

$\tilde{p} = \tilde{p}_1 + \tilde{p}_2$ where $\tilde{p}_1$ and $\tilde{p}_2$ solve the following wave equations:

$$\begin{cases} \Box \tilde{p}_1 = -\Delta \operatorname{div} \tilde{u} + \operatorname{div} \nabla p_0^{\varepsilon} \\ \tilde{p}_1(x,0) = \partial_\tau \tilde{p}_1(x,0) = 0 \\ \tilde{p}_1|_{\partial\Omega} = 0, \end{cases} \qquad \begin{cases} \Box \tilde{p}_2 = \operatorname{div} \left((\tilde{u} \cdot \nabla)\, \tilde{u} + \frac{1}{2}(\operatorname{div} \tilde{u})\tilde{u} \right) \\ \tilde{p}_2(x,0) = \partial_\tau \tilde{p}_2(x,0) = 0 \\ \tilde{p}_2|_{\partial\Omega} = 0. \end{cases}$$

$\tilde{p} = \tilde{p}_1 + \tilde{p}_2$ where $\tilde{p}_1$ and $\tilde{p}_2$ solve the following wave equations:

$$\begin{cases} \Box \tilde{p}_1 = -\Delta \operatorname{div} \tilde{u} + \operatorname{div} \nabla p_0^{\varepsilon} \\ \tilde{p}_1(x,0) = \partial_\tau \tilde{p}_1(x,0) = 0 \\ \tilde{p}_1|_{\partial\Omega} = 0, \end{cases} \qquad \begin{cases} \Box \tilde{p}_2 = \operatorname{div} \left((\tilde{u} \cdot \nabla) \, \tilde{u} + \frac{1}{2} (\operatorname{div} \tilde{u}) \tilde{u} \right) \\ \tilde{p}_2(x,0) = \partial_\tau \tilde{p}_2(x,0) = 0 \\ \tilde{p}_2|_{\partial\Omega} = 0. \end{cases}$$

we apply

$$\|w\|_{L^4_{t,x}} + \|\partial_t w\|_{L^4_t W^{-1,4}_x} \lesssim \|f\|_{\dot{H}^{1/2}_D} + \|g\|_{\dot{H}^{-1/2}_D} + \|F\|_{L^1_t L^2_x}$$

to $w = \Delta^{-1} \tilde{p}_1$

$$\|\tilde{p}_1\|_{L^4_\tau W^{-2,4}_x} + \|\partial_\tau \tilde{p}_1\|_{L^4_\tau W^{-3,4}_x} \lesssim \frac{\sqrt{T}}{\varepsilon^{1/4}} \|\operatorname{div} \tilde{u}\|_{L^2_\tau L^2_x} + \frac{T}{\varepsilon^{1/2}} \|p_0^\varepsilon\|_{L^2_x}$$

$\tilde p = \tilde p_1 + \tilde p_2$ where $\tilde p_1$ and $\tilde p_2$ solve the following wave equations:

$$\begin{cases} \Box \tilde p_1 = -\Delta \operatorname{div} \tilde u + \operatorname{div} \nabla p_0^\varepsilon \\ \tilde p_1(x,0) = \partial_\tau \tilde p_1(x,0) = 0 \\ \tilde p_1|_{\partial\Omega} = 0, \end{cases} \qquad \begin{cases} \Box \tilde p_2 = \operatorname{div} \left((\tilde u \cdot \nabla)\, \tilde u + \frac{1}{2} (\operatorname{div} \tilde u) \tilde u \right) \\ \tilde p_2(x,0) = \partial_\tau \tilde p_2(x,0) = 0 \\ \tilde p_2|_{\partial\Omega} = 0. \end{cases}$$

we apply

$$\|w\|_{L^4_{t,x}} + \|\partial_t w\|_{L^4_t W_x^{-1,4}} \lesssim \|f\|_{\dot H_D^{1/2}} + \|g\|_{\dot H_D^{-1/2}} + \|F\|_{L^1_t L_x^{3/2}}$$

to $w = \Delta^{-1/2} \tilde p_2$

$$\|\tilde p_2\|_{L^4_\tau W_x^{-1,4}} + \|\partial_\tau \tilde p_2\|_{L^4_\tau W_x^{-2,4}} \lesssim \| \left(\tilde u \cdot \nabla \right) \tilde u + 1/2 (\operatorname{div} \tilde u) \tilde u \|_{L^1_\tau L_x^{3/2}}$$

Estimate for the pressure

Finally we have the following estimate on p^ε

$$\varepsilon^{3/8}\|p^\varepsilon\|_{L^4_t W^{-2,4}_x} + \varepsilon^{7/8}\|\partial_t p^\varepsilon\|_{L^4_t W^{-3,4}_x} \lesssim T\|p^\varepsilon_0\|_{L^2_x} + \sqrt{T}\|\operatorname{div} u^\varepsilon\|_{L^2_t L^2_x}$$
$$+ \|\left(u^\varepsilon \cdot \nabla\right) u^\varepsilon + \frac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon\|_{L^1_t L^{3/2}_x}$$

Estimates on Qu^{ε} - Part 2

$$Qu^{\varepsilon} = \nabla \Delta_N^{-1} \operatorname{div} u^{\varepsilon}$$

Estimates on Qu^ε - Part 2

$$Qu^\varepsilon = \nabla \Delta_N^{-1} \operatorname{div} u^\varepsilon \qquad \varepsilon \partial_t p^\varepsilon = - \operatorname{div} u^\varepsilon$$

Estimates on Qu^ε - Part 2

$$Qu^\varepsilon = \nabla \Delta_N^{-1} \operatorname{div} u^\varepsilon \qquad \varepsilon \partial_t p^\varepsilon = -\operatorname{div} u^\varepsilon$$

$$Qu^\varepsilon = \varepsilon \nabla \Delta_N^{-1} \partial_t p^\varepsilon = \varepsilon^{1/8} \varepsilon^{7/8} \partial_t p^\varepsilon$$

Estimates on Qu^ε - Part 2

$$Qu^\varepsilon = \nabla\Delta_N^{-1}\operatorname{div} u^\varepsilon \qquad \varepsilon\partial_t p^\varepsilon = -\operatorname{div} u^\varepsilon$$

$$Qu^\varepsilon = \varepsilon\nabla\Delta_N^{-1}\partial_t p^\varepsilon = \varepsilon^{1/8}\varepsilon^{7/8}\partial_t p^\varepsilon$$

$$\varepsilon^{7/8}\|\partial_t p^\varepsilon\|_{L_t^4 W_x^{-3,4}}$$

Young-type estimates

$j \in C_0^\infty(\Omega)$, $j \geq 0$, $\int_\Omega j dx = 1$, $j_\alpha(x) = \alpha^{-d} j\left(\frac{x}{\alpha}\right)$.
Then the following Young type inequality hold

$$\|f * j_\alpha\|_{L^p(\Omega)} \leq C\alpha^{s-d\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{W^{-s,q}(\Omega)},$$

for any $p, q \in [1, \infty]$, $q \leq p$, $s \geq 0$, $\alpha \in (0, 1)$.

Young-type estimates

$j \in C_0^\infty(\Omega)$, $j \geq 0$, $\int_\Omega j dx = 1$, $j_\alpha(x) = \alpha^{-d} j\left(\frac{x}{\alpha}\right)$.
Then the following Young type inequality hold

$$\|f * j_\alpha\|_{L^p(\Omega)} \leq C\alpha^{s-d\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{W^{-s,q}(\Omega)},$$

for any $p, q \in [1, \infty]$, $q \leq p$, $s \geq 0$, $\alpha \in (0, 1)$.
Moreover for any $f \in \dot{H}^1(\Omega)$, one has

$$\|f - f * j_\alpha\|_{L^p(\Omega)} \leq C_p \alpha^{1-\sigma} \|\nabla f\|_{L^2(\Omega)},$$

where

$$p \in [2, \infty) \quad \text{if } d = 2, \quad p \in [2, 6] \quad \text{if } d = 3 \ \text{ and } \quad \sigma = d\left(\frac{1}{2} - \frac{1}{p}\right).$$

$$\|Qu^{\varepsilon}\|_{L_t^2 L_x^p} \leq \|Qu^{\varepsilon} * j_\alpha\|_{L_t^2 L_x^p} + \|Qu^{\varepsilon} - Qu^{\varepsilon} * j_\alpha\|_{L_t^2 L_x^p} = J_1 + J_2,$$

$$\|Qu^\varepsilon\|_{L^2_tL^p_x} \le \|Qu^\varepsilon * j_\alpha\|_{L^2_tL^p_x} + \|Qu^\varepsilon - Qu^\varepsilon * j_\alpha\|_{L^2_tL^p_x} = J_1 + J_2,$$

to estimate J_1 we use

$$\|f * j_\alpha\|_{L^p(\Omega)} \le C\alpha^{s-d\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{W^{-s,q}(\Omega)}, \text{ with } s = 2 \text{ and } q = 4$$

to get

$$J_1 \le \varepsilon^{1/8}\|\nabla\Delta_N^{-1}\varepsilon^{7/8}\partial_t p^\varepsilon * j\|_{L^2_tL^p_x} \le \varepsilon^{1/8}\alpha^{-2-3\left(\frac{1}{4}-\frac{1}{p}\right)}\|\varepsilon^{7/8}\partial_t p^\varepsilon\|_{L^2_tW^{-3,4}_x}$$

$$\le \varepsilon^{1/8}\alpha^{-2-3\left(\frac{1}{4}-\frac{1}{p}\right)}T^{1/4}\|\varepsilon^{7/8}\partial_t p^\varepsilon\|_{L^4_tW^{-3,4}_x}.$$

$$\|Qu^\varepsilon\|_{L^2_t L^p_x} \le \|Qu^\varepsilon * j_\alpha\|_{L^2_t L^p_x} + \|Qu^\varepsilon - Qu^\varepsilon * j_\alpha\|_{L^2_t L^p_x}$$
$$\le \varepsilon^{1/8} \alpha^{-2-3\left(\frac{1}{4}-\frac{1}{p}\right)} T^{1/4} \|\varepsilon^{7/8} \partial_t p^\varepsilon\|_{L^4_t W^{-3,4}_x} + J_2,$$

to estimate J_2 we use

$$\|f - f * j_\alpha\|_{L^p(\Omega)} \le C_p \alpha^{1-\sigma} \|\nabla f\|_{L^2(\Omega)},$$

to get

$$J_2 \le \alpha^{1-3\left(\frac{1}{2}-\frac{1}{p}\right)} \|Q\nabla u^\varepsilon\|_{L^2_t L^2_x}.$$

$$\begin{aligned}\|Qu^{\varepsilon}\|_{L^2_tL^p_x} &\leq \|Qu^{\varepsilon} * j_{\alpha}\|_{L^2_tL^p_x} + \|Qu^{\varepsilon} - Qu^{\varepsilon} * j_{\alpha}\|_{L^2_tL^p_x} \\ &\leq \varepsilon^{1/8}\alpha^{-2-3\left(\frac{1}{4}-\frac{1}{p}\right)}T^{1/4}\|\varepsilon^{7/8}\partial_t p^{\varepsilon}\|_{L^4_tW^{-3,4}_x} + \alpha^{1-3\left(\frac{1}{2}-\frac{1}{p}\right)}\|\nabla u^{\varepsilon}\|_{L^2_{t,x}} \\ &\leq C_T\varepsilon^{1/8}\alpha^{-2-3\left(\frac{1}{4}-\frac{1}{p}\right)} + C\alpha^{1-3\left(\frac{1}{2}-\frac{1}{p}\right)}\end{aligned}$$

$$
\begin{aligned}
\|Qu^{\varepsilon}\|_{L^2_t L^p_x} &\leq \|Qu^{\varepsilon} * j_\alpha\|_{L^2_t L^p_x} + \|Qu^{\varepsilon} - Qu^{\varepsilon} * j_\alpha\|_{L^2_t L^p_x} \\
&\leq \varepsilon^{1/8} \alpha^{-2-3\left(\frac{1}{4}-\frac{1}{p}\right)} T^{1/4} \|\varepsilon^{7/8} \partial_t p^{\varepsilon}\|_{L^4_t W^{-3,4}_x} + \alpha^{1-3\left(\frac{1}{2}-\frac{1}{p}\right)} \|\nabla u^{\varepsilon}\|_{L^2_{t,x}} \\
&\leq C_T \varepsilon^{1/8} \alpha^{-2-3\left(\frac{1}{4}-\frac{1}{p}\right)} + C\alpha^{1-3\left(\frac{1}{2}-\frac{1}{p}\right)}
\end{aligned}
$$

choose

$$
\alpha = \varepsilon^{1/18}
$$

$$
\begin{aligned}
\|Qu^{\varepsilon}\|_{L_t^2 L_x^p} &\le \|Qu^{\varepsilon} * j_\alpha\|_{L_t^2 L_x^p} + \|Qu^{\varepsilon} - Qu^{\varepsilon} * j_\alpha\|_{L_t^2 L_x^p} \\
&\le \varepsilon^{1/8} \alpha^{-2-3\left(\frac{1}{4}-\frac{1}{p}\right)} T^{1/4} \|\varepsilon^{7/8} \partial_t p^{\varepsilon}\|_{L_t^4 W_x^{-3,4}} + \alpha^{1-3\left(\frac{1}{2}-\frac{1}{p}\right)} \|\nabla u^{\varepsilon}\|_{L_{t,x}^2} \\
&\le C_T \varepsilon^{1/8} \alpha^{-2-3\left(\frac{1}{4}-\frac{1}{p}\right)} + C \alpha^{1-3\left(\frac{1}{2}-\frac{1}{p}\right)}
\end{aligned}
$$

choose

$$
\alpha = \varepsilon^{1/18}
$$

$$
\|Qu^{\varepsilon}\|_{L_t^2 L_x^p} \le C_T \varepsilon^{\frac{6-p}{36p}} \quad \text{for any } p \in [4,6).
$$

$$\begin{aligned}\|Qu^{\varepsilon}\|_{L^2_tL^p_x} &\le \|Qu^{\varepsilon} * j_\alpha\|_{L^2_tL^p_x} + \|Qu^{\varepsilon} - Qu^{\varepsilon} * j_\alpha\|_{L^2_tL^p_x} \\ &\le \varepsilon^{1/8}\alpha^{-2-3\left(\frac{1}{4}-\frac{1}{p}\right)}T^{1/4}\|\varepsilon^{7/8}\partial_t p^{\varepsilon}\|_{L^4_tW^{-3,4}_x} + \alpha^{1-3\left(\frac{1}{2}-\frac{1}{p}\right)}\|\nabla u^{\varepsilon}\|_{L^2_{t,x}} \\ &\le C_T\varepsilon^{1/8}\alpha^{-2-3\left(\frac{1}{4}-\frac{1}{p}\right)} + C\alpha^{1-3\left(\frac{1}{2}-\frac{1}{p}\right)}\end{aligned}$$

choose

$$\alpha = \varepsilon^{1/18}$$

$$\|Qu^{\varepsilon}\|_{L^2_tL^p_x} \le C_T\varepsilon^{\frac{6-p}{36p}} \quad \text{for any } p \in [4,6).$$

$$\Downarrow$$

$$Qu^{\varepsilon} \longrightarrow 0 \quad \text{strongly in } L^2_tL^p_x, \text{ for any } p \in [4,6).$$

Convergence on Pu^ε

- L^p compactness (Lions-Aubin theorem)

$$\|Pu^\varepsilon(t+h) - Pu^\varepsilon(t)\|_{L^2([0,T]\times\Omega)}$$

Convergence on Pu^ε

- L^p compactness (Lions-Aubin theorem)

$$\|Pu^\varepsilon(t+h) - Pu^\varepsilon(t)\|_{L^2([0,T]\times\Omega)}$$

$$\begin{aligned} Pz^\varepsilon = Pu^\varepsilon(t+h) - Pu^\varepsilon(t) &= \int_t^{t+h} \partial_s Pu^\varepsilon(s,x)ds \\ &= \int_t^{t+h} ds P(\Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \frac{1}{2} u^\varepsilon (\operatorname{div} u^\varepsilon))(s,x) \end{aligned}$$

Convergence on Pu^ε

- L^p compactness (Lions-Aubin theorem)

$$\|Pu^\varepsilon(t+h) - Pu^\varepsilon(t)\|_{L^2([0,T]\times\Omega)}$$

$$\begin{aligned} Pz^\varepsilon = Pu^\varepsilon(t+h) - Pu^\varepsilon(t) &= \int_t^{t+h} \partial_s Pu^\varepsilon(s,x)ds \\ &= \int_t^{t+h} dsP(\Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \frac{1}{2} u^\varepsilon (\operatorname{div} u^\varepsilon))(s,x) \end{aligned}$$

- convolution techniques

Convergence on Pu^ε

- L^p compactness (Lions-Aubin theorem)

$$\|Pu^\varepsilon(t+h) - Pu^\varepsilon(t)\|_{L^2([0,T]\times\Omega)}$$

$$Pz^\varepsilon = Pu^\varepsilon(t+h) - Pu^\varepsilon(t) = \int_t^{t+h} \partial_s Pu^\varepsilon(s,x)ds$$

$$= \int_t^{t+h} ds P(\Delta u^\varepsilon - (u^\varepsilon \cdot \nabla)\, u^\varepsilon - \frac{1}{2} u^\varepsilon (\operatorname{div} u^\varepsilon))(s,x)$$

- convolution techniques

$$\|Pu^\varepsilon(t+h) - Pu^\varepsilon(t)\|^2_{L^2_{t,x}} = \int_0^T \int_\Omega dtdx (Pz^\varepsilon) \cdot (Pz^\varepsilon - Pz^\varepsilon * j_\alpha)$$

$$+ \int_0^T \int_\Omega dtdx (Pz^\varepsilon) \cdot (Pz^\varepsilon * j_\alpha)$$

$$\|Pu^{\varepsilon}(t+h) - Pu^{\varepsilon}(t)\|^2_{L^2([0,T]\times\Omega)} \leq C(\alpha T^{1/2} + h\alpha^{-3/2}T^{1/2} + h)$$

$$\|Pu^{\varepsilon}(t+h) - Pu^{\varepsilon}(t)\|^2_{L^2([0,T]\times\Omega)} \leq C(\alpha T^{1/2} + h\alpha^{-3/2}T^{1/2} + h)$$

choose

$$\alpha = h^{2/5}$$

$$\|Pu^{\varepsilon}(t+h) - Pu^{\varepsilon}(t)\|^2_{L^2([0,T]\times\Omega)} \leq C(\alpha T^{1/2} + h\alpha^{-3/2}T^{1/2} + h)$$

choose

$$\alpha = h^{2/5}$$

$$\|Pu^{\varepsilon}(t+h) - Pu^{\varepsilon}(t)\|_{L^2([0,T]\times\Omega)} \leq C_T h^{1/5}$$

$$\|Pu^{\varepsilon}(t+h) - Pu^{\varepsilon}(t)\|^2_{L^2([0,T]\times\Omega)} \leq C(\alpha T^{1/2} + h\alpha^{-3/2}T^{1/2} + h)$$

choose

$$\alpha = h^{2/5}$$

$$\|Pu^{\varepsilon}(t+h) - Pu^{\varepsilon}(t)\|_{L^2([0,T]\times\Omega)} \leq C_T h^{1/5}$$

$$\Downarrow$$

$$Pu^{\varepsilon} \longrightarrow Pu, \qquad \text{strongly in } L^2(0,T;L^2_{loc}(\Omega))$$

Recover the pressure

$$Q\left(\partial_t u^{\varepsilon} + \nabla p^{\varepsilon} = \mu\Delta u^{\varepsilon} - \left(u^{\varepsilon}\cdot\nabla\right)u^{\varepsilon} - \frac{1}{2}(\operatorname{div} u^{\varepsilon})u^{\varepsilon}\right)$$

Recover the pressure

$$Q\left(\partial_t u^{\varepsilon} + \nabla p^{\varepsilon} = \mu\Delta u^{\varepsilon} - \left(u^{\varepsilon}\cdot\nabla\right)u^{\varepsilon} - \frac{1}{2}(\operatorname{div} u^{\varepsilon})u^{\varepsilon}\right)$$

$$\Downarrow$$

$$\nabla p^{\varepsilon} = Q\Delta u^{\varepsilon} - \partial_t Q u^{\varepsilon} - Q\left((u^{\varepsilon}\cdot\nabla)u^{\varepsilon}) + \frac{1}{2}u^{\varepsilon}\operatorname{div} Q u^{\varepsilon}\right).$$

Recover the pressure

$$Q\left(\partial_t u^\varepsilon + \nabla p^\varepsilon = \mu\Delta u^\varepsilon - (u^\varepsilon\cdot\nabla)\,u^\varepsilon - \frac{1}{2}(\operatorname{div} u^\varepsilon)u^\varepsilon\right)$$

$$\Downarrow$$

$$\nabla p^\varepsilon = Q\Delta u^\varepsilon - \partial_t Q u^\varepsilon - Q\left((u^\varepsilon\cdot\nabla)u^\varepsilon) + \frac{1}{2}u^\varepsilon \operatorname{div} Q u^\varepsilon\right).$$

$$\varepsilon \downarrow 0$$

$$\langle \nabla p^\varepsilon, \varphi\rangle \longrightarrow \langle \nabla \Delta_N^{-1}\operatorname{div}((u\cdot\nabla)u), \varphi\rangle \quad \text{for any } \varphi \in \mathcal{D}'([0,T]\times\Omega)$$

Recover the pressure

$$Q\left(\partial_t u^{\varepsilon} + \nabla p^{\varepsilon} = \mu\Delta u^{\varepsilon} - (u^{\varepsilon}\cdot\nabla)\,u^{\varepsilon} - \frac{1}{2}(\operatorname{div} u^{\varepsilon})u^{\varepsilon}\right)$$

$$\Downarrow$$

$$\nabla p^{\varepsilon} = Q\Delta u^{\varepsilon} - \partial_t Q u^{\varepsilon} - Q\left((u^{\varepsilon}\cdot\nabla)u^{\varepsilon}) + \frac{1}{2}u^{\varepsilon}\operatorname{div} Q u^{\varepsilon}\right).$$

$$\varepsilon \downarrow 0$$

$$\langle \nabla p^{\varepsilon}, \varphi\rangle \longrightarrow \langle \nabla\Delta_N^{-1}\operatorname{div}((u\cdot\nabla)u), \varphi\rangle \quad \text{for any } \varphi \in \mathcal{D}'([0,T]\times\Omega)$$

$$\Downarrow$$

$$p = \Delta_N^{-1}\operatorname{div}((u\cdot\nabla)u) = \Delta_N^{-1} tr((Du)^2)$$

Energy inequality

$$\int_\Omega \frac{1}{2}|u(x,t)|^2 dx + \int_0^T\int_\Omega |\nabla u(x,t)|^2 dxdt$$

$$\leq \liminf_{\varepsilon\to 0}\left(\int_\Omega \frac{1}{2}|u^\varepsilon(x,t)|^2 dx + \int_\Omega \frac{\varepsilon}{2}|p^\varepsilon|^2 + \int_0^T\int_\Omega |\nabla u^\varepsilon(x,t)|^2 dxdt\right)$$

$$= \liminf_{\varepsilon\to 0}\int_\Omega \frac{1}{2}\left(|u_0^\varepsilon|^2 - \varepsilon|p_0^\varepsilon|^2\right) dx = \int_\Omega \frac{1}{2}|u_0|^2 dx.$$

Where else the same phenomena appear?

Navier Stokes Poisson System in $\mathbb{R}^3$

is a simplified model to describe the dynamics of a plasma

$$\begin{cases} \partial_s \rho^\lambda + \operatorname{div}(\rho^\lambda u^\lambda) = 0 \\ \partial_s(\rho^\lambda u^\lambda) + \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda)) + \dfrac{\nabla(\rho^\lambda)^\gamma}{\gamma} = \overline{\mu}\Delta u^\lambda + (\overline{\mu} + \overline{\nu})\nabla \operatorname{div} u^\lambda + \rho^\lambda \nabla V^\lambda \\ \lambda^2 \Delta V^\lambda = \rho^\lambda - 1, \qquad x \in \mathbb{R}^3, s \geq 0 \end{cases}$$

Navier Stokes Poisson System in $\mathbb{R}^3$

is a simplified model to describe the dynamics of a plasma

$$\begin{cases} \partial_s \rho^\lambda + \mathrm{div}(\rho^\lambda u^\lambda) = 0 \\ \partial_s(\rho^\lambda u^\lambda) + \mathrm{div}(\rho^\lambda u^\lambda \otimes u^\lambda)) + \dfrac{\nabla(\rho^\lambda)^\gamma}{\gamma} = \overline{\mu}\Delta u^\lambda + (\overline{\mu} + \overline{\nu})\nabla \,\mathrm{div}\, u^\lambda + \rho^\lambda \nabla V^\lambda \\ \lambda^2 \Delta V^\lambda = \rho^\lambda - 1, \qquad x \in \mathbb{R}^3, s \geq 0 \end{cases}$$

$\rho^\lambda(x,t)$ is the *negative charge density*

$m^\lambda(x,t) = \rho^\lambda(x,t) u^\lambda(x,t)$ is the *current density*

$u^\lambda(x,t)$ is the *velocity vector density*

$V^\lambda(x,t)$ is the *electrostatic potential*

$\overline{\mu}$ is the *shear viscosity* and $\overline{\nu}$ is the *bulk viscosity*

λ is the so called *Debye length*

Navier Stokes Poisson System in $\mathbb{R}^3$

$$\begin{cases} \partial_s \rho^\lambda + \operatorname{div}(\rho^\lambda u^\lambda) = 0 \\ \partial_t(\rho^\lambda u^\lambda) + \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda)) + \dfrac{\nabla(\rho^\lambda)^\gamma}{\gamma} = \overline{\mu}\Delta u^\lambda + (\overline{\mu} + \overline{\nu})\nabla \operatorname{div} u^\lambda + \rho^\lambda \nabla V^\lambda \\ \lambda^2 \Delta V^\lambda = \rho^\lambda - 1 \end{cases}$$

Navier Stokes Poisson System in $\mathbb{R}^3$

$$\begin{cases} \partial_s \rho^\lambda + \mathrm{div}(\rho^\lambda u^\lambda) = 0 \\ \partial_t(\rho^\lambda u^\lambda) + \mathrm{div}(\rho^\lambda u^\lambda \otimes u^\lambda)) + \dfrac{\nabla(\rho^\lambda)^\gamma}{\gamma} = \overline{\mu}\Delta u^\lambda + (\overline{\mu} + \overline{\nu})\nabla \,\mathrm{div}\, u^\lambda + \rho^\lambda \nabla V^\lambda \\ \lambda^2 \Delta V^\lambda = \rho^\lambda - 1 \end{cases}$$

$\lambda =$ Debye lenght

$\downarrow 0$

Navier Stokes Poisson System in $\mathbb{R}^3$

$$\begin{cases} \partial_s \rho^\lambda + \operatorname{div}(\rho^\lambda u^\lambda) = 0 \\ \partial_t(\rho^\lambda u^\lambda) + \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda)) + \dfrac{\nabla(\rho^\lambda)^\gamma}{\gamma} = \overline{\mu}\Delta u^\lambda + (\overline{\mu} + \overline{\nu})\nabla \operatorname{div} u^\lambda + \rho^\lambda \nabla V^\lambda \\ \lambda^2 \Delta V^\lambda = \rho^\lambda - 1 \end{cases}$$

$\lambda =$ Debye lenght

$$\downarrow 0$$

Incompressible Navier Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases}$$

Physical background

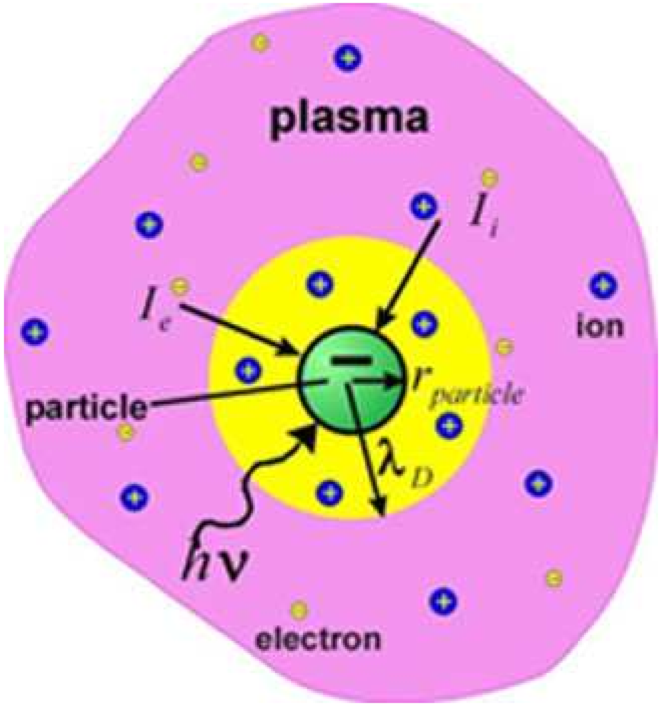
a charged particle inside a plasma attracts particles with opposite charge and repels those with the same charge

Physical background

a charged particle inside a plasma attracts particles with opposite charge and repels those with the same charge

$\Downarrow$

creation of a net cloud of opposite charge around itself



the particle's Coulomb field fall off as e^{-r}, rather than as $1/r^2$

$$V(r) = \frac{q e^{-r/\lambda}}{4\pi\epsilon_0 r} \qquad \lambda = \sqrt{\frac{\epsilon_0 k T}{2 n_0 e^2}}$$

$$V(r) = \frac{qe^{-r/\lambda}}{4\pi\epsilon_0 r} \qquad \lambda = \sqrt{\frac{\epsilon_0 kT}{2n_0 e^2}}$$

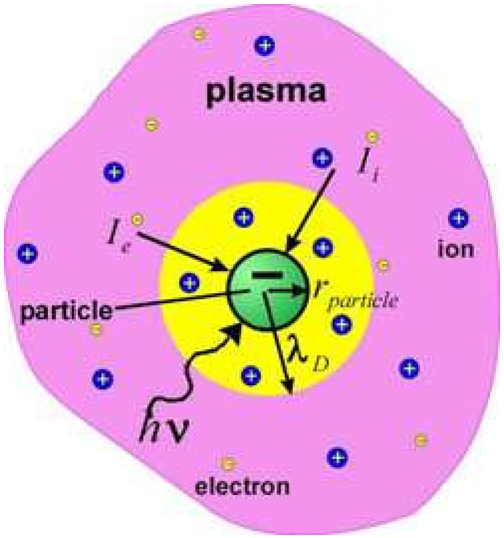
$\Rightarrow$ the plasma beyond a distance λ is essentially shielded from the effects of the charge

$\Rightarrow$ we don't expect to find electric fields existing in a plasma over regions greater in extend than λ

$$V(r) = \frac{qe^{-r/\lambda}}{4\pi\epsilon_0 r} \qquad \lambda = \sqrt{\frac{\epsilon_0 kT}{2n_0e^2}}$$

⇒ the plasma beyond a distance λ is essentially shielded from the effects of the charge

⇒ we don't expect to find electric fields existing in a plasma over regions greater in extend than λ



ϵ_0 = dielectric constant $= 8,85 \cdot 10^{-12} C^2/m^2 N$

k = Boltzmann constant $= 1,38 \cdot 10^{-23} Nm/K$

T = electron temperature $= 10^4 K$

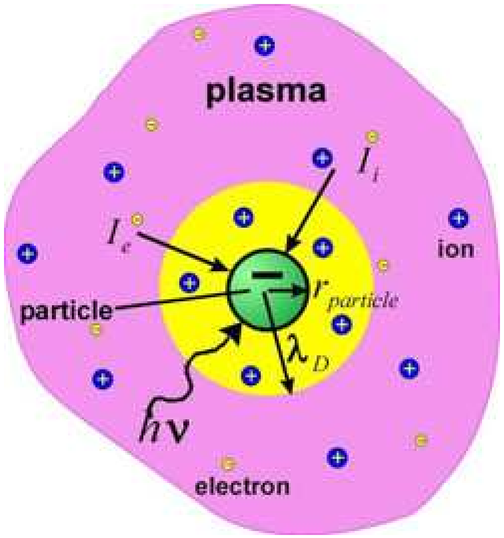
n_0 = electron density $= 10^{16} m^{-3}$

e = electron charge $= -1,6 \cdot 10^{-19} C$

$$V(r) = \frac{qe^{-r/\lambda}}{4\pi\epsilon_0 r} \qquad \lambda = \sqrt{\frac{\epsilon_0 kT}{2n_0 e^2}}$$

$\Rightarrow$ the plasma beyond a distance λ is essentially shielded from the effects of the charge

$\Rightarrow$ we don't expect to find electric fields existing in a plasma over regions greater in extend than λ



ϵ_0 = dielectric constant $= 8,85 \cdot 10^{-12} C^2/m^2 N$

k = Boltzmann constant $= 1,38 \cdot 10^{-23} Nm/K$

T = electron temperature $= 10^4 K$

n_0 = electron density $= 10^{16} m^{-3}$

e = electron charge $= -1,6 \cdot 10^{-19} C$

$$\lambda \approx 10^{-4} m$$

Scaling

$$\begin{cases} \partial_s \rho^\lambda + \operatorname{div}(\rho^\lambda u^\lambda) = 0 \\ \partial_t(\rho^\lambda u^\lambda) + \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda)) + \dfrac{\nabla(\rho^\lambda)^\gamma}{\gamma} = \overline{\mu}\Delta u^\lambda + (\overline{\mu} + \overline{\nu})\nabla \operatorname{div} u^\lambda + \rho^\lambda \nabla V^\lambda \\ \lambda^2 \Delta V^\lambda = \rho^\lambda - 1 \end{cases}$$

Scaling

$$\begin{cases} \partial_s \rho^\lambda + \operatorname{div}(\rho^\lambda u^\lambda) = 0 \\ \partial_t(\rho^\lambda u^\lambda) + \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda)) + \dfrac{\nabla(\rho^\lambda)^\gamma}{\gamma} = \overline{\mu}\Delta u^\lambda + (\overline{\mu} + \overline{\nu})\nabla \operatorname{div} u^\lambda + \rho^\lambda \nabla V^\lambda \\ \lambda^2 \Delta V^\lambda = \rho^\lambda - 1 \end{cases}$$

Time scaling: $s = \dfrac{t}{\varepsilon}$ $\qquad \overline{\mu} = \varepsilon\mu, \overline{\nu} = \varepsilon\nu$

$$\rho^\varepsilon(x,t) = \rho^\lambda\left(x, \frac{t}{\varepsilon}\right),\ u^\varepsilon = \frac{1}{\varepsilon}u^\lambda\left(x, \frac{t}{\varepsilon}\right),\ V^\varepsilon = V^\lambda\left(x, \frac{t}{\varepsilon}\right).$$

Scaling

$$\begin{cases} \partial_s \rho^\lambda + \operatorname{div}(\rho^\lambda u^\lambda) = 0 \\ \partial_t(\rho^\lambda u^\lambda) + \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda)) + \dfrac{\nabla(\rho^\lambda)^\gamma}{\gamma} = \overline{\mu}\Delta u^\lambda + (\overline{\mu} + \overline{\nu})\nabla \operatorname{div} u^\lambda + \rho^\lambda \nabla V^\lambda \\ \lambda^2 \Delta V^\lambda = \rho^\lambda - 1 \end{cases}$$

Time scaling: $s = \dfrac{t}{\varepsilon}$ $\qquad \overline{\mu} = \varepsilon\mu, \overline{\nu} = \varepsilon\nu$

$$\rho^\varepsilon(x,t) = \rho^\lambda\left(x, \frac{t}{\varepsilon}\right), \; u^\varepsilon = \frac{1}{\varepsilon} u^\lambda\left(x, \frac{t}{\varepsilon}\right), \; V^\varepsilon = V^\lambda\left(x, \frac{t}{\varepsilon}\right).$$

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0 \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \dfrac{\nabla(\rho^\varepsilon)^\gamma}{\gamma\varepsilon^2} = \mu\Delta u^\varepsilon + (\nu + \mu)\nabla \operatorname{div} u^\varepsilon + \dfrac{\rho^\varepsilon}{\varepsilon^2}\nabla V^\varepsilon \\ \lambda^2 \Delta V^\varepsilon = \rho^\varepsilon - 1. \end{cases}$$

Scaling

$$\begin{cases} \partial_s \rho^\lambda + \operatorname{div}(\rho^\lambda u^\lambda) = 0 \\ \partial_t(\rho^\lambda u^\lambda) + \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda)) + \dfrac{\nabla(\rho^\lambda)^\gamma}{\gamma} = \overline{\mu}\Delta u^\lambda + (\overline{\mu} + \overline{\nu})\nabla \operatorname{div} u^\lambda + \rho^\lambda \nabla V^\lambda \\ \lambda^2 \Delta V^\lambda = \rho^\lambda - 1 \end{cases}$$

Time scaling: $s = \dfrac{t}{\varepsilon}$ $\qquad \overline{\mu} = \varepsilon\mu, \overline{\nu} = \varepsilon\nu$

$$\rho^\varepsilon(x,t) = \rho^\lambda\left(x, \frac{t}{\varepsilon}\right), \; u^\varepsilon = \frac{1}{\varepsilon} u^\lambda\left(x, \frac{t}{\varepsilon}\right), \; V^\varepsilon = V^\lambda\left(x, \frac{t}{\varepsilon}\right).$$

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0 \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \dfrac{\nabla(\rho^\varepsilon)^\gamma}{\gamma\varepsilon^2} = \mu\Delta u^\varepsilon + (\nu + \mu)\nabla \operatorname{div} u^\varepsilon + \dfrac{\rho^\varepsilon}{\varepsilon^2}\nabla V^\varepsilon \\ \lambda^2 \Delta V^\varepsilon = \rho^\varepsilon - 1. \end{cases}$$

$\varepsilon^\beta = \lambda^2,$ where $\beta > 0$

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0 \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \dfrac{\nabla(\rho^\varepsilon)^\gamma}{\gamma \varepsilon^2} = \mu \Delta u^\varepsilon + (\nu + \mu) \nabla \operatorname{div} u^\varepsilon + \dfrac{\rho^\varepsilon}{\varepsilon^2} \nabla V^\varepsilon \\ \varepsilon^\beta \Delta V^\varepsilon = \rho^\varepsilon - 1. \end{cases}$$

renormalized pressure:

$$\pi^\varepsilon = \frac{(\rho^\varepsilon)^\gamma - 1 - \gamma(\rho^\varepsilon - 1)}{\varepsilon^2 \gamma(\gamma - 1)}$$

density fluctuation:

$$\sigma^\varepsilon = \frac{\rho^\varepsilon - 1}{\varepsilon}$$

$$
\begin{cases}
\partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0 \\
\partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \dfrac{\nabla(\rho^\varepsilon)^\gamma}{\gamma \varepsilon^2} = \mu \Delta u^\varepsilon + (\nu + \mu)\nabla \operatorname{div} u^\varepsilon + \dfrac{\rho^\varepsilon}{\varepsilon^2}\nabla V^\varepsilon \\
\varepsilon^\beta \Delta V^\varepsilon = \rho^\varepsilon - 1.
\end{cases}
$$

renormalized pressure: **density fluctuation:**

$$\pi^\varepsilon = \frac{(\rho^\varepsilon)^\gamma - 1 - \gamma(\rho^\varepsilon - 1)}{\varepsilon^2 \gamma(\gamma - 1)} \qquad \sigma^\varepsilon = \frac{\rho^\varepsilon - 1}{\varepsilon}$$

initial conditions:

$$\int_{\mathbb{R}^3} \left(\pi^\varepsilon|_{t=0} + \frac{|m_0^\varepsilon|^2}{2\rho_0^\varepsilon} + \varepsilon^{\beta-2}|V_0^\varepsilon|^2 \right) dx \le C_0, \quad \text{where}$$

$$\rho^\varepsilon u^\varepsilon|_{t=0} = m_0^\varepsilon \quad m_0^\varepsilon dx \ll \sqrt{\rho_0^\varepsilon} dx \qquad \frac{m_0^\varepsilon}{\sqrt{\rho_0^\varepsilon}} \rightharpoonup u_0 \text{ weakly in } L^2(\mathbb{R}^3)$$

A priori estimate

$$\int_{\mathbb{R}^3}\left(\rho^{\varepsilon}\frac{|u^{\varepsilon}|^2}{2}+\frac{(\rho^{\varepsilon})^{\gamma}-1-\gamma(\rho^{\varepsilon}-1)}{\varepsilon^2\gamma(\gamma-1)}+\varepsilon^{\beta-2}|\nabla V^{\varepsilon}|^2\right)dx$$
$$+\int_0^t\int_{\mathbb{R}^3}\left(\mu|\nabla u^{\varepsilon}|^2+(\nu+\mu)|\operatorname{div} u^{\varepsilon}|^2\right)dxds\leq C_0.$$

A priori estimate

$$\int_{\mathbb{R}^3}\left(\rho^\varepsilon\frac{|u^\varepsilon|^2}{2}+\frac{(\rho^\varepsilon)^\gamma-1-\gamma(\rho^\varepsilon-1)}{\varepsilon^2\gamma(\gamma-1)}+\varepsilon^{\beta-2}|\nabla V^\varepsilon|^2\right)dx$$
$$+\int_0^t\int_{\mathbb{R}^3}\left(\mu|\nabla u^\varepsilon|^2+(\nu+\mu)|\operatorname{div}u^\varepsilon|^2\right)dxds\le C_0.$$

$$\Downarrow$$

A priori estimate

$$\int_{\mathbb{R}^3}\left(\rho^\varepsilon\frac{|u^\varepsilon|^2}{2}+\frac{(\rho^\varepsilon)^\gamma-1-\gamma(\rho^\varepsilon-1)}{\varepsilon^2\gamma(\gamma-1)}+\varepsilon^{\beta-2}|\nabla V^\varepsilon|^2\right)dx$$
$$+\int_0^t\int_{\mathbb{R}^3}\left(\mu|\nabla u^\varepsilon|^2+(\nu+\mu)|\operatorname{div}u^\varepsilon|^2\right)dxds\le C_0.$$

$$\Downarrow$$

∇u^ε is bounded in $L^2_{t,x}$, $\varepsilon^{\frac{\beta}{2}-1}\nabla V^\varepsilon$ is bounded in $L^\infty_t L^2_x$,

A priori estimate

$$\int_{\mathbb{R}^3}\left(\rho^{\varepsilon}\frac{|u^{\varepsilon}|^2}{2}+\frac{(\rho^{\varepsilon})^{\gamma}-1-\gamma(\rho^{\varepsilon}-1)}{\varepsilon^2\gamma(\gamma-1)}+\varepsilon^{\beta-2}|\nabla V^{\varepsilon}|^2\right)dx$$

$$+\int_0^t\int_{\mathbb{R}^3}\left(\mu|\nabla u^{\varepsilon}|^2+(\nu+\mu)|\operatorname{div}u^{\varepsilon}|^2\right)dxds\leq C_0.$$

$$\Downarrow$$

∇u^{ε} is bounded in $L^2_{t,x}$, $\varepsilon^{\frac{\beta}{2}-1}\nabla V^{\varepsilon}$ is bounded in $L^{\infty}_t L^2_x$,

u^{ε} is bounded in $L^2_{t,x}\cap L^2_t L^6_x$ $\sigma^{\varepsilon}u^{\varepsilon}$ is bounded in $L^2_t H^{-1}_x$

Estimate on Qu^ε

Since

$$\rho^\varepsilon = \varepsilon\sigma^\varepsilon + 1$$

we have

Estimate on Qu^ε

Since

$$\rho^\varepsilon = \varepsilon\sigma^\varepsilon + 1$$

we have

$$Qu^\varepsilon = Q(\rho^\varepsilon u^\varepsilon) - \varepsilon Q(\sigma^\varepsilon u^\varepsilon)$$

Estimate on Qu^ε

Since

$$\rho^\varepsilon = \varepsilon\sigma^\varepsilon + 1$$

we have

$$Qu^\varepsilon = Q(\rho^\varepsilon u^\varepsilon) - \varepsilon Q\underbrace{(\sigma^\varepsilon u^\varepsilon)}_{L^2_t H^{-1}_x}$$

Estimate on Qu^ε

Since

$$\rho^\varepsilon = \varepsilon\sigma^\varepsilon + 1$$

we have

$$Qu^\varepsilon = Q\underbrace{(\rho^\varepsilon u^\varepsilon)}_{\to 0 \quad ?} - \varepsilon Q\underbrace{(\sigma^\varepsilon u^\varepsilon)}_{L_t^2 H_x^{-1}}$$

but......

Estimate on Qu^ε

Since

$$\rho^\varepsilon = \varepsilon\sigma^\varepsilon + 1$$

we have

$$Qu^\varepsilon = Q\underbrace{(\rho^\varepsilon u^\varepsilon)}_{\to 0 \quad ?} - \varepsilon Q\underbrace{(\sigma^\varepsilon u^\varepsilon)}_{L^2_t H^{-1}_x}$$

but......

$$Q(\rho^\varepsilon u^\varepsilon) = \nabla\Delta^{-1}\operatorname{div}(\rho^\varepsilon u^\varepsilon)$$

Estimate on Qu^ε

Since

$$\rho^\varepsilon = \varepsilon\sigma^\varepsilon + 1$$

we have

$$Qu^\varepsilon = Q\underbrace{(\rho^\varepsilon u^\varepsilon)}_{\to 0 \quad ?} - \varepsilon Q\underbrace{(\sigma^\varepsilon u^\varepsilon)}_{L^2_t H^{-1}_x}$$

but......

$$Q(\rho^\varepsilon u^\varepsilon) = \nabla\Delta^{-1}\operatorname{div}(\rho^\varepsilon u^\varepsilon) \qquad \varepsilon\partial_t\sigma^\varepsilon = -\operatorname{div}(\rho^\varepsilon u^\varepsilon)$$

Estimate on Qu^ε

Since

$$\rho^\varepsilon = \varepsilon\sigma^\varepsilon + 1$$

we have

$$Qu^\varepsilon = Q\underbrace{(\rho^\varepsilon u^\varepsilon)}_{\to 0\quad ?} - \varepsilon Q\underbrace{(\sigma^\varepsilon u^\varepsilon)}_{L^2_t H^{-1}_x}$$

but......

$$Q(\rho^\varepsilon u^\varepsilon) = \nabla\Delta^{-1}\operatorname{div}(\rho^\varepsilon u^\varepsilon) \qquad \varepsilon\partial_t\sigma^\varepsilon = -\operatorname{div}(\rho^\varepsilon u^\varepsilon)$$

$$Q(\rho^\varepsilon u^\varepsilon) = \varepsilon\nabla\Delta^{-1}\partial_t\sigma^\varepsilon$$

Density fluctuation wave equation

$$
\begin{aligned}
\partial_t \sigma^\varepsilon + \frac{1}{\varepsilon}\operatorname{div}(\rho^\varepsilon u^\varepsilon) &= 0\\
\partial_t(\rho^\varepsilon u^\varepsilon) + \frac{1}{\varepsilon}\nabla\sigma^\varepsilon &= \mu\Delta u^\varepsilon + (\nu+\mu)\nabla\operatorname{div} u^\varepsilon - \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon)\\
&\quad - (\gamma-1)\nabla\pi^\varepsilon + \frac{\sigma^\varepsilon}{\varepsilon}\nabla V^\varepsilon + \frac{1}{\varepsilon^2}\nabla V^\varepsilon,\\
\varepsilon^{\beta-1}\Delta V^\varepsilon &= \sigma^\varepsilon.
\end{aligned}
$$

Density fluctuation wave equation

$$\partial_t \sigma^\varepsilon + \frac{1}{\varepsilon} \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0$$

$$\partial_t(\rho^\varepsilon u^\varepsilon) + \frac{1}{\varepsilon}\nabla \sigma^\varepsilon = \mu \Delta u^\varepsilon + (\nu + \mu)\nabla \operatorname{div} u^\varepsilon - \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon)$$

$$- (\gamma - 1)\nabla \pi^\varepsilon + \frac{\sigma^\varepsilon}{\varepsilon}\nabla V^\varepsilon + \frac{1}{\varepsilon^2}\nabla V^\varepsilon,$$

$$\varepsilon^{\beta-1}\Delta V^\varepsilon = \sigma^\varepsilon.$$

Differentiate in t the "density fluctuation equation", taking the divergence of the second equation

$$\varepsilon^2 \partial_{tt}\sigma^\varepsilon - \Delta \sigma^\varepsilon = -\varepsilon^2 \operatorname{div}(\mu \Delta u^\varepsilon + (\nu + \mu)\nabla \operatorname{div} u^\varepsilon)$$

$$+ \varepsilon^2 \operatorname{div} \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \varepsilon^2 (\gamma - 1) \operatorname{div} \nabla \pi^\varepsilon$$

$$- \varepsilon \operatorname{div}(\sigma^\varepsilon \nabla V^\varepsilon) - \operatorname{div} \nabla V^\varepsilon$$

changing the time scale: $t = \varepsilon\tau$

$$\tilde{\sigma}(x,\tau) = \sigma^{\varepsilon}(x,\varepsilon\tau), \qquad \tilde{\rho}(x,\tau) = \rho^{\varepsilon}(x,\varepsilon\tau)$$

$$\tilde{u}(x,\tau) = u^{\varepsilon}(x,\varepsilon\tau), \qquad \tilde{V}(x,\tau) = V^{\varepsilon}(x,\varepsilon\tau)$$

changing the time scale: $t = \varepsilon\tau$

$$\tilde{\sigma}(x,\tau) = \sigma^{\varepsilon}(x,\varepsilon\tau), \qquad \tilde{\rho}(x,\tau) = \rho^{\varepsilon}(x,\varepsilon\tau)$$

$$\tilde{u}(x,\tau) = u^{\varepsilon}(x,\varepsilon\tau), \qquad \tilde{V}(x,\tau) = V^{\varepsilon}(x,\varepsilon\tau)$$

$$\begin{aligned}\partial_{\tau\tau}\tilde{\sigma} - \Delta\tilde{\sigma} = &-\varepsilon^2 \operatorname{div}(\mu\Delta\tilde{u} + (\nu+\mu)\nabla \operatorname{div}\tilde{u}) \\ &+ \varepsilon^2 \operatorname{div}(\operatorname{div}(\tilde{\rho}\tilde{u}\otimes\tilde{u}) + (\gamma-1)\nabla\tilde{\pi}) \\ &- \varepsilon \operatorname{div}(\tilde{\sigma}\nabla\tilde{V}) - \operatorname{div}\nabla\tilde{V}.\end{aligned}$$

changing the time scale: $t = \varepsilon\tau$

$$\tilde{\sigma}(x,\tau) = \sigma^{\varepsilon}(x,\varepsilon\tau), \qquad \tilde{\rho}(x,\tau) = \rho^{\varepsilon}(x,\varepsilon\tau)$$

$$\tilde{u}(x,\tau) = u^{\varepsilon}(x,\varepsilon\tau), \qquad \tilde{V}(x,\tau) = V^{\varepsilon}(x,\varepsilon\tau)$$

$$\partial_{\tau\tau}\tilde{\sigma} - \Delta\tilde{\sigma} = -\varepsilon^2 \operatorname{div} \underbrace{(\mu\Delta\tilde{u} + (\nu+\mu)\nabla \operatorname{div} \tilde{u})}_{L^2_t H^{-1}_x}$$

$$+\varepsilon^2 \operatorname{div}(\operatorname{div} \underbrace{(\tilde{\rho}\tilde{u}\otimes\tilde{u})}_{L^\infty_t L^1_x} + (\gamma-1)\nabla \underbrace{\tilde{\pi}}_{L^\infty_t L^1_x})$$

$$-\varepsilon \operatorname{div} \underbrace{(\tilde{\sigma}\nabla\tilde{V})}_{L^\infty_t L^1_x} - \operatorname{div} \underbrace{\nabla\tilde{V}}_{L^\infty_t L^2_x}.$$

$\tilde{\sigma}=\tilde{\sigma}_1 + \tilde{\sigma}_2 + \tilde{\sigma}_3 + \tilde{\sigma}_4$ where $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$ and $\tilde{\sigma}_4$ solve the systems:

$\tilde\sigma = \tilde\sigma_1 + \tilde\sigma_2 + \tilde\sigma_3 + \tilde\sigma_4$ where $\tilde\sigma_1$, $\tilde\sigma_2$, $\tilde\sigma_3$ and $\tilde\sigma_4$ solve the systems:

$$\begin{cases} \Box\tilde\sigma_1 = -\Delta \operatorname{div} \tilde u = \varepsilon^2 F_1 \\ \tilde\sigma_1(x,0) = \tilde\sigma_0 \ \partial_\tau\tilde\sigma_1(x,0) = \partial_\tau\tilde\sigma_0, \end{cases} \qquad \begin{cases} \Box\tilde\sigma_2 = \varepsilon^2 F_2 \\ \tilde\sigma_2(x,0) = \partial_\tau\tilde\sigma_2(x,0) = 0. \end{cases}$$

$$\begin{cases} \Box\tilde\sigma_3 = \varepsilon F_3 \\ \tilde\sigma_3(x,0) = \partial_\tau\tilde\sigma_3(x,0) = 0. \end{cases} \qquad \begin{cases} \Box\tilde\sigma_4 = F_4 \\ \tilde\sigma_4(x,0) = \partial_\tau\tilde\sigma_4(x,0) = 0. \end{cases}$$

$\tilde{\sigma}=\tilde{\sigma}_1+\tilde{\sigma}_2+\tilde{\sigma}_3+\tilde{\sigma}_4$ where $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$ and $\tilde{\sigma}_4$ solve the systems:

$$\begin{cases}\Box\tilde{\sigma}_1 = -\Delta\,\mathrm{div}\,\tilde{u} = \varepsilon^2 F_1\\ \tilde{\sigma}_1(x,0) = \tilde{\sigma}_0\ \partial_\tau\tilde{\sigma}_1(x,0) = \partial_\tau\tilde{\sigma}_0,\end{cases} \qquad \begin{cases}\Box\tilde{\sigma}_2 = \varepsilon^2 F_2\\ \tilde{\sigma}_2(x,0) = \partial_\tau\tilde{\sigma}_2(x,0) = 0.\end{cases}$$

$$\begin{cases}\Box\tilde{\sigma}_3 = \varepsilon F_3\\ \tilde{\sigma}_3(x,0) = \partial_\tau\tilde{\sigma}_3(x,0) = 0.\end{cases} \qquad \begin{cases}\Box\tilde{\sigma}_4 = F_4\\ \tilde{\sigma}_4(x,0) = \partial_\tau\tilde{\sigma}_4(x,0) = 0.\end{cases}$$

Finally we have the following estimate on σ^ε

$$\begin{aligned}&\varepsilon^{-\frac{1}{4}+\frac{\beta}{2}}\|\sigma^\varepsilon\|_{L^4_t W_x^{-s_0-2,4}} + \varepsilon^{\frac{3}{4}+\frac{\beta}{2}}\|\partial_t\sigma^\varepsilon\|_{L^4_t W_x^{-s_0-3,4}}\\ &\lesssim \varepsilon^{\frac{\beta}{2}}\|\sigma_0^\varepsilon\|_{H_x^{-1}} + \varepsilon^{\frac{\beta}{2}}\|m_0^\varepsilon\|_{H_x^{-1}}\\ &+\varepsilon^{1+\frac{\beta}{2}}T\|\,\mathrm{div}(\mathrm{div}(\sigma^\varepsilon u^\varepsilon\otimes u^\varepsilon)-(\gamma-1)\nabla\pi^\varepsilon)\|_{L_t^\infty H_x^{-s_0-2}}\\ &+\varepsilon^{1+\frac{\beta}{2}}\|\,\mathrm{div}\,\Delta u^\varepsilon + \nabla\,\mathrm{div}\,u^\varepsilon\|_{L_t^2 H_x^{-2}}\\ &+T\|\,\mathrm{div}\,\nabla V^\varepsilon\|_{L_t^\infty H_x^{-1}} + \varepsilon^{1+\frac{\beta}{2}}T\|\varepsilon^{\beta-2}\,\mathrm{div}(\sigma^\varepsilon V^\varepsilon)\|_{L_t^\infty H_x^{-s_0-1}}\end{aligned}$$

Estimates on Qu^{ε}- Part 2

Estimates on Qu^{ε}- Part 2

$$Q(\rho^{\varepsilon}u^{\varepsilon}) = \nabla\Delta^{-1}\,\mathrm{div}(\rho^{\varepsilon}u^{\varepsilon}) \qquad \varepsilon\partial_t\sigma^{\varepsilon} = -\,\mathrm{div}(\rho^{\varepsilon}u^{\varepsilon})$$

$$Q(\rho^{\varepsilon}u^{\varepsilon}) = \varepsilon\nabla\Delta^{-1}\partial_t\sigma^{\varepsilon}$$

Estimates on Qu^ε- Part 2

$$Q(\rho^\varepsilon u^\varepsilon) = \nabla\Delta^{-1}\,\mathrm{div}(\rho^\varepsilon u^\varepsilon) \qquad\qquad \varepsilon\partial_t\sigma^\varepsilon = -\,\mathrm{div}(\rho^\varepsilon u^\varepsilon)$$

$$Q(\rho^\varepsilon u^\varepsilon) = \varepsilon^{\frac{1}{4}-\frac{\beta}{2}}\nabla\Delta^{-1}\varepsilon^{\frac{3}{4}+\frac{\beta}{2}}\partial_t\sigma^\varepsilon$$

$$\varepsilon^{\frac{3}{4}+\frac{\beta}{2}}\|\partial_t\sigma^\varepsilon\|_{L^4_t W^{-s_0-3,4}_x}$$

Estimates on Qu^ε- Part 2

$$Q(\rho^\varepsilon u^\varepsilon) = \nabla\Delta^{-1}\operatorname{div}(\rho^\varepsilon u^\varepsilon) \qquad \varepsilon\partial_t\sigma^\varepsilon = -\operatorname{div}(\rho^\varepsilon u^\varepsilon)$$

$$Q(\rho^\varepsilon u^\varepsilon) = \varepsilon^{\frac{1}{4}-\frac{\beta}{2}}\nabla\Delta^{-1}\varepsilon^{\frac{3}{4}+\frac{\beta}{2}}\partial_t\sigma^\varepsilon$$

$$\varepsilon^{\frac{3}{4}+\frac{\beta}{2}}\|\partial_t\sigma^\varepsilon\|_{L^4_t W_x^{-s_0-3,4}}$$

$$\frac{1}{4}-\frac{\beta}{2} > 0 \qquad \text{if} \quad \beta \le \frac{1}{2}$$

Estimates on Qu^ε- Part 2

$$Q(\rho^\varepsilon u^\varepsilon) = \nabla\Delta^{-1}\,\mathrm{div}(\rho^\varepsilon u^\varepsilon) \qquad \varepsilon\partial_t\sigma^\varepsilon = -\,\mathrm{div}(\rho^\varepsilon u^\varepsilon)$$

$$Q(\rho^\varepsilon u^\varepsilon) = \varepsilon^{\frac{1}{4}-\frac{\beta}{2}}\nabla\Delta^{-1}\varepsilon^{\frac{3}{4}+\frac{\beta}{2}}\partial_t\sigma^\varepsilon$$

$$\varepsilon^{\frac{3}{4}+\frac{\beta}{2}}\|\partial_t\sigma^\varepsilon\|_{L^4_t W_x^{-s_0-3,4}}$$

$$\frac{1}{4}-\frac{\beta}{2} > 0 \qquad \text{if} \quad \beta \leq \frac{1}{2}$$

this analysis holds for physical regimes of order

$$M = \varepsilon = \lambda^{2/\beta} \approx 10^{-16}$$

Extensions

Extensions

- Model for plasma physics that takes into account the temperature effects and balance equation

Extensions

- Model for plasma physics that takes into account the temperature effects and balance equation
- Bipolar models for semiconductors